Differential Geometry

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These notes accompany the module *Differential Geometry* for third-year undergraduates reading mathematics at the University of Hull.

We will delve into fundamental properties of curves and surfaces, largely in 3D. Many results from Vector Calculus, Analysis and Linear Algebra will be assumed.

The main textbook for this course is

> *Elementary Differential Geometry*, Pressley A., 2nd ed., Springer (2010).

This is available as an e-book from BJL. In addition, I recommend the following books, also in BJL.

> *Differential Geometry of Curves and Surfaces*, do Carmo M. P., Prentice-Hall (1976).
>
> *Elementary Differential Geometry*, Bär C., Cambridge University Press (2010).
>
> *Differential Geometry of Curves and Surfaces*, Banchoff T. and Lovett S., 2nd ed., CRC Press (2015).

Although MATLAB is not required for this course, you might find it helpful for plotting curves and surfaces that would otherwise be difficult to visualise. If you don't have MATLAB on your computer – see this page:
`https://share.hull.ac.uk/Services/ICT/Home%20use%20software/Matlab.aspx`.

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SC
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How to plot curves and surfaces in MATLAB?

- To plot a parametric curve in $\mathbb{R}^2$, read the help file on `fplot`
- To plot a parametric curve in $\mathbb{R}^3$, read the help file on `fplot3`
- To plot a parametric surface in $\mathbb{R}^3$, read the help file on `fsurf`

Chapter 1

Curves

Our starting point is curves in $\mathbb{R}^2$ and $\mathbb{R}^3$, described in the language of *vectors*.

1.1 Parametrized curves

Definition. A *parametrized curve* in $\mathbb{R}^n$ is a vector-valued function

$$= \tag{1.1}$$

where each component $\gamma_i(t)$ is a real function, and ______________ (some subset of $\mathbb{R}$).

We always take I to be an *open* set (this will be justified later). I will use a bold $\boldsymbol{\gamma}$ to denote a curve (vector). We will also assume that all curves in this course can be differentiated *infinitely many times* on I, in other words, each γ_i is a _____ function on I.

It gets tiring to say "$\boldsymbol{\gamma}$ is infinitely differentiable", so instead, we say "$\boldsymbol{\gamma}$ is _______________". Normally, smooth means C^1, but only in this course do we take this word to mean C^∞.

Even if a function looks a bit suspicious, *e.g.* $\boldsymbol{\gamma}(t) = \big(\ln(t), \sqrt{t}\big)$, we will work with the interval of I for which $\boldsymbol{\gamma}$ is smooth (in this case, $I =$ __________).

Example 1. Write down the parametric equation of a line-segment joining two points with position vectors $\mathbf{a}$ and $\mathbf{b}$.

Example 2. (a) Find a vector parametrization of the curve $y = x^2$.

(b) Is $\boldsymbol{\gamma} = (t^2, t^4)$, $t \in \mathbb{R}$, a good parametrization?

Example 3. Write down a vector parametrization of
a) the unit circle, b) the *upper* unit semi-circle.

Example 4. Sketch the curve (in $\mathbb{R}^3$): $\boldsymbol{\gamma}(t) = (\cos t, \sin t, t)$, $t \in (0, 4\pi)$.

Example 5. Find the Cartesian equation $\boldsymbol{\gamma}(t) = (e^t + 1, t^2)$, $t \in \mathbb{R}$. Sketch the curve.

Next we have a familiar result from Vector Calculus (consult your old notes).

Proposition 1.1. The tangent vector to the curve $\boldsymbol{\gamma}(t)$ is __________.

Example 6. Find the tangent vector to the curve in Example 5 at $t = 0$.

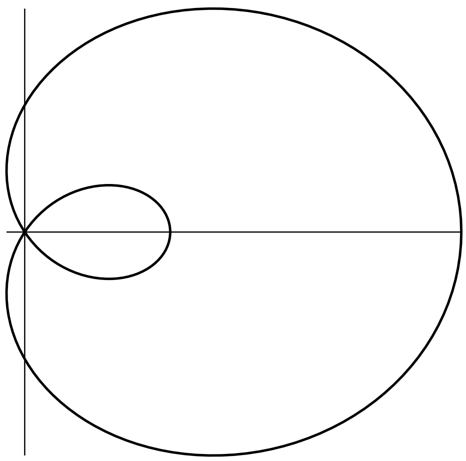
Example 7. A curve $\boldsymbol{\gamma}(t)$ has a constant tangent vector for all t. What curve is this? (PS. vector functions can be integrated and differentiated as usual – component-wise.)

Parametric curves can often intersect itself. This means that a single point on the curve can be specified by more than one value of t.

Example 8. Find the values of t for which the following curve[1] passes through the origin.

$$\boldsymbol{\gamma}(t) = \begin{pmatrix} (1 + 2\cos t)\cos t \\ (1 + 2\cos t)\sin t \end{pmatrix}, t \in (0, 2\pi).$$

Find the tangent vectors at those values of t.



[1]This is the *Limaçon of Pascal*. It is the locus traced out by a point on a circle rolling on another circle. See some nice graphics here: `http://bit.ly/1MZqifR`

1.2 Arc-length parametrization

Proposition 1.2. The *arc-length* of the curve $\boldsymbol{\gamma}(t)$ between $t = t_0$ and $t = t_1$ is given by

$$= \qquad (1.2)$$

See the Vector Calculus for proof, but we can roughly see why this is true as follows:

Definition. The arc-length *function* is defined as

$$= \qquad \text{for some constant } t_0. \qquad (1.3)$$

Important: (1.2) is a number, but (1.3) is a function.

If we think about $\boldsymbol{\gamma}(t)$ physically as the trajectory (displacement) of a particle. Then the vector $\dot{\boldsymbol{\gamma}}(t)$ is just its ______________, and the magnitude $|\dot{\boldsymbol{\gamma}}(t)|$ is its ______________. The vector $\ddot{\boldsymbol{\gamma}}(t)$ is its ___________________.

Definition. The scalar function $|\dot{\boldsymbol{\gamma}}(t)|$ is called the *speed* of the curve $\boldsymbol{\gamma}(t)$.

Example 9. (a) Calculate the length of the helix which winds round a cylinder of unit radius *once*.

(b) Obtain the arc-length function, $s(t)$, with $t = 0$ at the starting point.

(c) Write down the parametrization of the helix using s instead of t. Find the speed of the curve in this parametrization.

Definition. The *arc-length parametrization* of $\boldsymbol{\gamma}(t)$ is denoted $\tilde{\boldsymbol{\gamma}}(s)$, where s is the __________________________ (with some initial parameter value t_0).

Lemma 1.3. Let $\tilde{\boldsymbol{\gamma}}(s)$ be the arc-length parametrization of a curve $\boldsymbol{\gamma}(t)$. Suppose that $|\dot{\boldsymbol{\gamma}}(t)| \neq 0$, then $\tilde{\boldsymbol{\gamma}}(s)$ has unit speed.
Proof:

(We'll deal with the converse later.) It's important to keep in mind that both $\boldsymbol{\gamma}(t)$ and $\tilde{\boldsymbol{\gamma}}(s)$ describe the *same* curve - just parametrized differently. Where there is no confusion, we can write $\boldsymbol{\gamma}(s)$ instead of $\tilde{\boldsymbol{\gamma}}(s)$.

Example 10. Find the arc-length parametrization of the semicircle radius $R > 0$:

$$\{(x, y) : x^2 + y^2 = R^2,\ y > 0\}.$$

Example 11. Find the arc-length parametrization of the curve:

$$\boldsymbol{\gamma}(t) = (t, t^2, t^3),\ t > 0.$$

The moral of the story: it's often not possible to find the arc-length parametrization explicitly.

But then, why do we bother with arc-length parametrization if it's so hard to calculate? Later on, we will see that it simplifies many otherwise nasty formulae and proofs. Most of the time we won't need to know the explicit form of the arc-length parametrization, $s(t)$, but just the fact that it *exists* is enough.

1.3 Reparametrization

Let's generalise the idea of using a different parameter to describe the curve $\boldsymbol{\gamma}(t)$ as $\tilde{\boldsymbol{\gamma}}(s)$, where s is not necessarily the arc-length parameter.

Definition. A function $\phi : A \to B$ is said to be a ________________________ if:

- ϕ is smooth
- ϕ is invertible
- ϕ^{-1} is also smooth

If such a ϕ exists, the two sets A and B are said to be ____________________.

Definition. A curve $\tilde{\boldsymbol{\gamma}}(s)$ is said to be a ______________________ of $\boldsymbol{\gamma}(t)$ if there is diffeomorphism ϕ such that $\phi : S \to T$ (S and T are some subsets of $\mathbb{R}$) with

$$= \tag{1.4}$$

Important: The two curves are exactly the same (same image, same geometry) – only the points on it are described differently. The mapping on the right explains this situation. This kind of diagram will be very useful throughout the course.

Not any random parametrization is a *reparametrization*: we must check that ϕ is indeed a diffeomorphism.

Example 12. Show that $\tilde{\boldsymbol{\gamma}}(s) = \Big(R\sin s, R\cos s\Big)$ is a reparametrization of the semicircle $\boldsymbol{\gamma}(t) = \Big(R\cos t, R\sin t\Big)$ in Example 10. Identify the diffeomorphism ϕ in this case.

Important: The function $t = \phi(s)$ can also be written (in short) as __________.
(Analogy: $y = f(x)$ can be written as $y(x)$ on a lazy day.)
Similarly, $s = \phi^{-1}(t)$ can be written as __________.
Similarly for derivatives: we can either deal with $\dot{\phi}(s) = d\phi/ds$ or __________.
(Analogy: $f'(x)$, df/dx and dy/dx all refer to the same thing). Be flexible.

Lemma 1.4. Let ϕ be a smooth bijection. Then the derivatives of ϕ and ϕ^{-1} are nonzero.
Proof:

Definitions. The curve $\boldsymbol{\gamma}(t)$ is said to be a *regular curve* if

Question. Find a regular and a non-regular parametrization for the line $y = x$.

This means that "regularity" is a property of parametrizations - and not the geometry of a curve. You can't tell whether a curve is regular by just looking at its shape.

Lemma 1.5. A reparametrization of a regular curve is again regular.
Proof:

The next theorem requires a little result from Analysis.

Lemma 1.6. Let f be a strictly increasing, smooth function in $[a, b]$, then f^{-1} exists, is smooth, and increasing on $[f(a), f(b)]$.

See any textbook on real analysis for a simple proof. Actually, only the fact that $f' \neq 0$ is enough for f to be invertible – this is the *inverse function theorem*. More about this later, but it's worth looking it up if you haven't heard of it before.

Lemma 1.7. Let $\boldsymbol{\gamma}(t)$ be the parametrization of a regular curve. Then, its arc-length function, $s(t)$, is a strictly increasing, smooth function.
Proof:

Theorem 1.8. A curve has a unit-speed reparametrization *iff* it is regular.
Proof: We will prove the '*if*' part. The '*only if*' part is easier (homework).

The big conclusion which we've just shown is: a unit-speed reparametrization *exists* for all regular curves. But is it unique?

Corollary. Let $\boldsymbol{\gamma}(u)$ and $\tilde{\boldsymbol{\gamma}}(s)$ be two unit-speed reparametrizations of a curve, then

$$u = \pm s + C, \tag{1.5}$$

where C is a constant.
Proof:

The previous Corollary implies that all unit-speed parametrizations are *effectively* the arc-length parametrization, but possibly with

(a) a shift in the parameter values. (b) increasing or decreasing parameters.

(or a combination of both. More about (b) in sheet 1, Q4). Let's think about this pictorially. Suppose we have a curve of length 10 units, what are some possible unit-speed parametrizations?

Whatever unit-speed parametrization you choose, it's always describing the same curve with length 10 units.

So from now on, we can use the terms "*arc-length parametrization*" and "*unit-speed parametrization*" interchangeably, keeping in mind these degrees of freedom.

In the next Chapter, we look at further properties of curves, including the concept of the *Frenet Frame*. I recommend revisiting Chapter 1 of your Vector Calculus notes.

Example 13. (a) Find an open interval I on which the curve

$$\boldsymbol{\gamma}(t) = (\cos^2 t, \sin^2 t), \quad t \in I,$$

is a regular curve.

(b) Find a unit-speed parametrization of the curve.

(c) Sketch the curve.

Chapter 2

Curvature and Torsion

In this Chapter, we will focus on two important characteristics of curves in $\mathbb{R}^3$: the curvature, which quantifies how much a curve curves, and the torsion, which measures the deviation of a curve from a 2D plane.

2.1 Curvature

So *how much does a curve curve?* A huge circle seems to curve __________ than a little circle (the outer lane of a big running track is mostly straight). We want to come up with a quantity which measures this *curvature*: let's call it κ. If R is the radius of a circle, then already we should expect

$$\propto \tag{2.1}$$

Definition. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve. The *curvature* is defined as the function

$$=$$

Important: The curvature is a function (scalar) – not a vector. Also, this definition only works for unit-speed parametrizations - we will deal with the general case later.

Now let's see if this definition of curvature is consistent with what we expect.

Example 1. Calculate the curvature of a circle centred at the origin with radius R.

Example 2. Prove that two unit-speed parametrizations of a curve have the same curvature.

This isn't surprising: the geometry of a curve is unchanged by reparametrizations.

Before we proceed, here are some useful results from vector calculus.

Warning: *These won't be given in the exam.*

(a) $\mathbf{a} \cdot \mathbf{b} =$

(b) $|\mathbf{a} \times \mathbf{b}| =$

(c) $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} =$

(d) $\mathbf{a} \cdot \mathbf{a} \times \mathbf{c} =$

(e) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) =$

(f) $|\mathbf{a} \times (\mathbf{a} \times \mathbf{b})| =$

(g) $\dfrac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) =$

(h) $\dfrac{d}{dt}(\mathbf{a} \times \mathbf{b}) =$

Lemma 2.1. If $|\dot{\boldsymbol{\gamma}}|$ =constant, prove that $\dot{\boldsymbol{\gamma}}$ and $\ddot{\boldsymbol{\gamma}}$ are orthogonal.

Lemma 2.1 looks very simple, but it will be very useful for the rest of this Chapter.

Theorem 2.2. The curvature of a regular curve, $\boldsymbol{\gamma}(t)$, is given by

$$\kappa(t) = \frac{|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''|}{|\boldsymbol{\gamma}'|^3},$$

where a dash indicates a derivative wrt t.

Example 3. What curve is this?

$$\boldsymbol{\gamma}(t) = (a\cos t, a\sin t, bt), \ t \in \mathbb{R}. \tag{2.2}$$

Calculate its curvature. Discuss the cases when i) $a = 0$ $(b \neq 0)$, ii) $b = 0$ $(a \neq 0)$.

2.2 Curvature in 2D

In this section, let's assume that the curves are 2 dimensional, *i.e.* $\boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t))$. Whilst $\kappa \geq 0$, in 2D we should be able to say more about the *sign* of the curvature (*e.g.* $y = x^2$ VS $y = -x^2$).

Definition. The *unit tangent* to the curve $\boldsymbol{\gamma}(t)$ is defined as the vector

$$\mathbf{t} =$$

If $\boldsymbol{\gamma}(s)$ is a unit-speed curve, then clearly $\mathbf{t} =$ __________.

When we discuss curves with parameter s or t, it helps to distinguish between the two derivatives. From now on, I will use a dash for t and a dot for s derivative.

Definition. The *signed unit normal*, $\mathbf{n}_s$, is defined as the vector obtained by rotating the unit tangent by 90° anticlockwise, *i.e.* if $\mathbf{t} = (t_1, t_2)$, then

$$\mathbf{n}_s =$$

Note that $\mathbf{t} \cdot \mathbf{n}_s =$ _____.

Question. Why does Lemma 2.1 imply that $\ddot{\boldsymbol{\gamma}}$ and $\mathbf{n}_s$ are multiple of each other?

Definition. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve. The *signed curvature* is the scalar κ_s such that

$$\ddot{\boldsymbol{\gamma}} = \kappa_s \mathbf{n}_s. \tag{2.3}$$

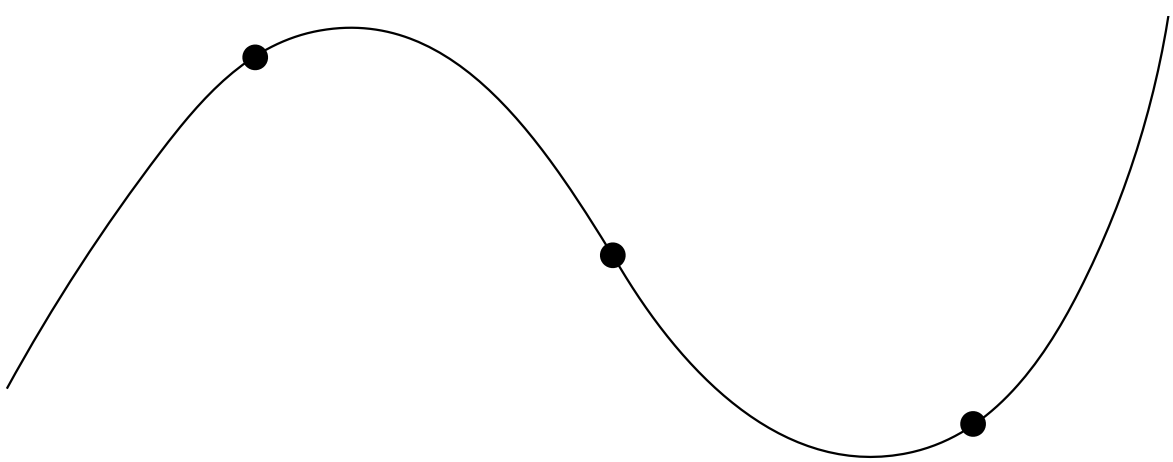
The curvature κ is always non-negative, but κ_s can change sign. For curves in $\mathbb{R}^2$, both quantities exist, but for curves in $\mathbb{R}^3$ we cannot talk about κ_s (why?).

Example 4. Show that for a unit-speed curve in $\mathbb{R}^2$, $\boldsymbol{\gamma}(s)$, we have $\kappa = |\kappa_s|$.

Now we will argue that the concept of positive/negative curvature in the plane coincides with the 'concave up/down' concept from school maths.

Study the diagram of a plane curve $\boldsymbol{\gamma}(s)$ below. Assume that the curve is swept out from left to right as s increases from 0. Mark in the directions of the unit tangents, signed normals, and $\ddot{\boldsymbol{\gamma}}$ at the marked points.

Note: It might help to think of the derivative of a vector as the change in the vector as it moves to a *nearby* point.



You can now see that the point where $\kappa_s > 0$ (*i.e.* where $\mathbf{n}_s$ and $\ddot{\boldsymbol{\gamma}}$ point in the same direction) is concave up (also called positive *concavity*), and similar for negative curvature/concavity. At $\kappa_s = 0$ we have a point of _______________, around which $\mathbf{t}$ is roughly constant. So the signed curvature is simply the generalisation of concavity.

The next Example shows what happens to Eq. 2.3 when $\boldsymbol{\gamma}$ is not unit speed.

Example 5. Let $\boldsymbol{\gamma}(t)$ be a regular curve and let $\boldsymbol{\gamma}(s)$ be its unit-speed parametrization. Let s' denote $\mathrm{d}s/\mathrm{d}t$. Deduce that $\boldsymbol{\gamma}' = s'\dot{\boldsymbol{\gamma}}$, and find a similar expression for $\boldsymbol{\gamma}''$.

Next, let's study an alternative interpretation of the signed curvature as the rate of *rotation* of the tangent vector $\mathbf{t}$.

Definition. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve in 2D. The *turning angle*, φ, is the angle that the tangent vector makes relative to the x axis, *i.e.*

$$\mathbf{t} = \dot{\boldsymbol{\gamma}}(s) = \tag{2.4}$$

Example 6. Roughly sketch the graph of $\varphi(s)$ for the curve segment on the previous page. Is this graph unique?

We can regard φ as a reparametrization $s \to \varphi(s)$. One can actually show that the turning-angle function, $\varphi(s)$, such as the one you sketched, exists for any unit-speed parametrized curve. Furthermore, $\varphi(s)$ will be smooth and unique (up multiples of 2π).[1]

[1] For an easy proof, see Bär, p. 37.

Lemma 2.3. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve in 2D with turning angle $\varphi(s)$. Then,

$$\kappa_s = \frac{\mathrm{d}\varphi}{\mathrm{d}s}. \tag{2.5}$$

Example 7. Find the arc-length function $s(t)$ for the curve $\boldsymbol{\gamma}(t) = (t, \cosh t)$ $(t > 0)$. Hence, use (2.5) to calculate $\kappa_s(s)$ and $\kappa_s(t)$.

(See problem sheet 2 for yet another way to do calculate κ_s.)

Before we move on to curvature in $\mathbb{R}^3$, we should mention a very special theorem: It turns out that the signed curvature completely characterises plane curves (up to rotation and translation). See Pressley for proof.

Theorem 2.4. [Fundamental Theorem of Plane Curves] Let $\kappa_s \colon I \to \mathbb{R}$ be any given smooth function $(I \subseteq \mathbb{R})$.

[Existence] There exists a unit-speed curve $\boldsymbol{\gamma} : I \to \mathbb{R}^2$ whose signed curvature is κ_s.

[Uniqueness] If $\tilde{\boldsymbol{\gamma}}$ is another unit-speed curve with signed curvature κ_s, then $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are related by a sequence of rotations and translations in $\mathbb{R}^2$.

2.3 Curvature in $\mathbb{R}^3$

We've just shown that in $\mathbb{R}^2$, the signed curvature κ_s essentially determines the shape of the curve. In $\mathbb{R}^3$, however, the curvature $\kappa =$ __________ alone is not enough to completely characterise a curve. Another property is required to quantify how much a curve is 'lifted' out of the plane: this is the *torsion*, as we will see shortly.

Definition. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve in $\mathbb{R}^3$, we define the *principal normal* as the unit vector $\mathbf{n}$ such that

$$\ddot{\boldsymbol{\gamma}} = \kappa\,\mathbf{n}. \tag{2.6}$$

Compare this with Eq. 2.3 for curves in $\mathbb{R}^2$.

We can assume that $\kappa \not\equiv 0$ (otherwise, see Sheet 2 Q1).

Question. Can you see why $\mathbf{n}$ is indeed a unit vector?

As before, we will use the notation $\mathbf{t} =$ _____ for the unit tangent. This means that (2.6) can also be written as

$$= \tag{2.7}$$

Example 8. Prove that $\mathbf{n}$ is perpendicular to $\mathbf{t}$.

Definition. The *binormal* of a unit-speed curve is defined as the vector

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}. \tag{2.8}$$

Clearly $\mathbf{b}$ is perpendicular to both $\mathbf{t}$ and $\mathbf{n}$, so $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ forms an __________________________.

Example 9. Find $|\mathbf{b}|$ and $\mathbf{n} \times \mathbf{b}$.

Thus we can perform calculations with $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ in the same way as we do with $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

Definition. The basis $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is called the *Frenet*[2] *frame* for space curves.

Question. What is the equivalent of the $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ frame for plane curves?

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[2] *Jean Frédéric Frenet* (1816-1900), French mathematician. Sometimes also attributed to *Joseph Serret* (1819-1885), another French mathematician who independently studied space curves in this formulation.

Example 10. Prove that $\dot{\mathbf{b}}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{t}$.

This Example showed that $\dot{\mathbf{b}}$ is parallel to _____. Let $-\tau$ be the proportionality 'constant':

$$\dot{\mathbf{b}} = -\tau\mathbf{n}. \tag{2.9}$$

(the minus sign is just a convention). Let's think about what τ means. If $\tau = 0$, then $\mathbf{b}$ is a constant vector. This means that the curve stays in the ____________ spanned by $\mathbf{t}$ and $\mathbf{n}$. We will prove this properly later, but that's the idea.

Definition. The *torsion* of a unit-speed curve with nonzero curvature is τ defined in Eq. 2.9.

Example 11. Prove that two unit-speed parametrizations of a curve $\boldsymbol{\gamma}(t)$ have the same curvature and torsion.

Solution: Let $\boldsymbol{\gamma}(s)$ and $\boldsymbol{\gamma}(u)$ be two unit-speed parametrizations of $\boldsymbol{\gamma}(t)$. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the basis calculated wrt to s, and $\{\tilde{\mathbf{t}}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}\}$ that wrt to u. From Chapter 1, we know that u and s are related by _______________. It's sufficient to consider the case when $u = -s$ (the 2 curves have opposite orientations).

Recall that every regular curve has a unit-speed reparametrization. This means that we can associate κ and τ to every regular curve. Thus, we have deduced the following:

Corollary. The curvature κ and torsion τ of a regular curve are both unchanged by reparametrizations.

In other words, κ and τ are fundamental attributes of a regular curve, independent of its parametrization. How does τ look like for curves that are not necessarily unit speed? In problem sheet 2, you'll show that:

Theorem 2.5. Let $\boldsymbol{\gamma}(t)$ be a regular 3D curve on which $\kappa \neq 0$. The torsion is given by

$$\tau = \frac{\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'' \cdot \boldsymbol{\gamma}'''}{|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''|^2} \tag{2.10}$$

We now show that τ quantifies how much a curve escapes from a 2D plane.

Theorem 2.6. Let $\boldsymbol{\gamma}$ be a regular 3D curve on which $\kappa \neq 0$. The curve is contained in a plane *iff* $\tau = 0$ on the entire curve.

Proof. WLOG, let us work with a unit-speed parametrization $\boldsymbol{\gamma}(s)$ of the curve.

Theorem 2.7. [*Frenet-Serret equations*] Let $\boldsymbol{\gamma}$ be a 3D unit-speed curve on which $\kappa \neq 0$. We have the relations

$$\begin{bmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \\ \dot{\mathbf{b}} \end{bmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{2.11}$$

Proof. Eqs. 2.7 and 2.9 provide the first and third lines. To prove the second line, differentiate the cyclic relation, $\mathbf{n} = \mathbf{b} \times \mathbf{t}$.

Example 12. For the helix $\boldsymbol{\gamma}(t) = (\cos t, \sin t, t)$ $(t > 0)$, find the Frenet basis $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, as well as its curvature and torsion. Work with a unit-speed parametrization.

Example 13. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve. Express $\dddot{\boldsymbol{\gamma}}$ in the Frenet basis. Hence show that

$$\dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}} \times \dddot{\boldsymbol{\gamma}} = \kappa^2 \tau.$$

Example 14. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve with constant curvature $\kappa > 0$ and $\tau = 0$.

(a) Show that $\boldsymbol{\gamma} + \kappa^{-1}\mathbf{n}$ is a constant vector.

(b) Hence prove that $\boldsymbol{\gamma}$ is part of a circle. Find the radius of the circle.

This is why the quantity $1/\kappa$ is called the radius of curvature (regardless of whether $\boldsymbol{\gamma}$ is a circle or not). It locally quantifies the amount that the curve curves, by comparing it with a 'best-approximating' circle (in 2D) or sphere (in 3D). These are called the *osculating* circle or sphere of $\boldsymbol{\gamma}$.

A couple of concluding remarks to close this Chapter.

- The fact that an anti-symmetric matrix appears in the Frenet-Serret equations is not a coincidence. The reason is found in the fields of *Lie group* and *Lie algebra*...

- In 2D, the signed curvature κ_s specifies curves up to translation and rotation. We have a similar situation in 3D:

  Theorem 2.8. [Fundamental Theorem of Space Curves] Let $\kappa : I \to \mathbb{R}^+$ and $\tau : I \to \mathbb{R}$ be smooth functions ($I \subseteq \mathbb{R}$).

  [Existence] There exists a unit-speed curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ with curvature κ and torsion τ.

  [Uniqueness] If $\tilde{\boldsymbol{\gamma}}$ is another unit-speed curve with curvature κ and torsion τ, then $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are related by a sequence of rotation and translations in $\mathbb{R}^3$.

- Calculating κ and τ for a given curve $\boldsymbol{\gamma}$ can be a messy business without a plan. Here's a helpful flowchart.

Chapter 3

Global properties of plane curves

We now study *global* properties of curves in $\mathbb{R}^2$, meaning that we are no longer interested in their local properties like κ or τ at a certain point, but rather their *macroscopic* properties, *i.e.* properties of the curve as a whole. For example, we will prove that the area enclosed by a closed curve of a given length is maximised *iff* it is a circle.

We will only deal with closed curves in $\mathbb{R}^2$ in this Chapter.

We are going to study 3 big theorems connected with closed curves, namely, 1) Hopf's Umlaufsatz, 2) The Isoperimetric Inequality, and 3) The Four-Vertex Theorem.

Definition. A curve $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^2$ is said to be ______________ if there exists a finite constant $L > 0$ such that, for all $t \in \mathbb{R}$,

$$\boldsymbol{\gamma}(t + L) = \boldsymbol{\gamma}(t) \tag{3.1}$$

The *smallest* such L is its ____________. We can also say that the curve is _______________.

Definition. A smooth curve $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^2$ is said to be __________ if it is regular and periodic.

Example 1. Show that $\boldsymbol{\gamma}(t) = (\cos t, \sin t)$, where $t \in \mathbb{R}$, is a closed curve.

The definition of a closed curve is not as straightforward as you might first think. Can you tell whether this curve is closed?

$$\boldsymbol{\gamma}(t) = (\cos t, \cos^2 t), \quad t \in \mathbb{R}.$$

Here's another example. This curve looks 'closed' but it is not classified as such.

Since a closed curve is regular, we can work with its arc-length parametrization, which can be shown to be periodic, and the period equals its ______________. Though this sounds completely obvious, the proof is slightly technical though not difficult (see Pressley page 21). We will state it here as a lemma.

Lemma 3.1. If $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^2$ is a closed curve with length ℓ, then its arc-length parametrization, $\tilde{\boldsymbol{\gamma}}(s)$, is also a closed curve with period L, where

$$=$$

Definition. A closed curve $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^2$ is __________ if it has no self-intersection.

Equivalently, a curve $\boldsymbol{\gamma}$ is simple if and only if the *function* $\boldsymbol{\gamma}$ is ______________.

Question. Sketch a closed curve which is simple and one which is not simple.

3.1 Hopf's *Umlaufsatz*

This section concerns the 'total' signed curvature as you go round a closed curve, *i.e.*

(3.2)

(we can just work with the arc-length parametrization). Why is this interesting? Well, note that for unit-speed curves, we can write the signed curvature as the derivative of the turning angle (Lemma 2.3), so that

(3.3)

where L is the length of the closed curve. So this integral actually measures the total turning angle which the tangent sweeps out as the curves comes back onto itself.

In the case of the circle, you might be able to guess the answer.

Example 2. Calculate $\oint \kappa_s \, \mathrm{d}s$ for $\boldsymbol{\gamma}(t) = (R\cos t, R\sin t)$, where $t \in \mathbb{R}$.

Can you think of a closed curve which winds around *more than once* before closing up? In this case, you can imagine that $\oint \kappa_s \, \mathrm{d}s > 2\pi$.

Lemma 3.2. Let $\boldsymbol{\gamma}$ be a unit-speed closed curve, then the total signed curvature is a multiple of _____, *i.e.*

$$\oint \kappa_s \, \mathrm{d}s = \qquad\qquad \text{for some } N \in \mathbb{Z}$$

Proof. Again let's work with the turning angle and arc-length parametrized $\boldsymbol{\gamma}$. Let L be the period. As before, this means:

$$\oint \kappa_s \, \mathrm{d}s = \tag{3.4}$$

and we want to show that this is a multiple of 2π.

Since $\boldsymbol{\gamma}(s)$ is unit speed, its period is equal to the arc-length L (by Lemma 3.1). This means that $\boldsymbol{\gamma}(s+L) =$ __________ for all s. Clearly, the unit tangent, $\mathbf{t}$, is also L-periodic, and, in particular, $\mathbf{t}(s=0) = \mathbf{t}(s=L)$ (otherwise $\boldsymbol{\gamma}$ would not be smooth).

In terms of the turning angle, $\mathbf{t}(s) = (\cos\varphi(s), \sin\varphi(s))$, so

This implies that $\varphi(L) - \varphi(0)$ is a multiple of 2π. □

Definition. The integer N in Lemma 3.2 is called the ____________________ of $\boldsymbol{\gamma}$.

Intuitively, N is the number of times the curve goes around itself before closing up.

Example 3. Draw closed curves with rotation index $N = 0, \pm 1, \pm 2, 3$.

You might have noticed that other than the cases $N = \pm 1$, the curves have self-intersections, *i.e.* they are not simple. This key observation is precisely...

Theorem 3.3. [*Hopf's Umlaufsatz*] The rotation index of a simple closed curve in $\mathbb{R}^2$ is either ________________, *i.e.*

$$\oint \kappa_s \, \mathrm{d}s = \pm 2\pi.$$

In German, *Umlauf* means rotation or circulation, and *Satz* means theorem. It is named after the German mathematician Heinz Hopf (1894-1971).

The proof is rather long and technical and we won't go into it in this course[1], however, you are expected to be able to state what the theorem says. You should have an intuitive feel of what it means.

Question. How can you tell whether a curve has rotation index 1 or -1?

Definition. A simple closed curve is said to be __________________________ if it has rotation index _____. Otherwise it is *negatively oriented.*

Example 4. Let $\boldsymbol{\gamma}(t)$ be a circle radius 2 centred at the origin. Find $\displaystyle\oint \kappa_s(t) \, \mathrm{d}t$. Comment on the answer in light of the Umlaufsatz.

[1]See Bär, page 41.

Example 5. By applying Hopf's Umlaufsatz to the curve $\boldsymbol{\gamma}(t) = (p\cos t, q\sin t)$, (where $p > 0$ and $q > 0$), evaluate the integral

$$\int_0^{2\pi} \frac{\mathrm{d}t}{p^2\sin^2 t + q^2\cos^2 t}.$$

3.2 Isoperimetric Inequality

What is the maximum area that can be formed with a given length of rope?

This is the famous *isoperimetric problem.* Our intuition suggests that the solution is a circle, but proving it is surprisingly difficult. The earliest attempt was probably by Zenodoros (200BC[2]) but a rigorous proof was not established until the 19th century. Here we study an interesting proof based on differential geometry.

Phrasing the problem in terms of calculus, we are interested in the area inside a *simple* closed curve $\boldsymbol{\gamma}$. We denote this area by the double integral

$$A = \tag{3.5}$$

where Ω is the finite area bounded by $\boldsymbol{\gamma}$.
Here are a few results we will need for the proof. Remember this from Vector Calculus?

Theorem 3.4. [__________ *Theorem*] Let Ω be the region in the plane bounded by a simple closed curve $\boldsymbol{\gamma}$, which is positively oriented. Let $P(x, y)$ and $Q(x, y)$ be functions with continuous 1st-order partial derivatives in Ω, then

$$\iint_{\Omega} (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y =$$

Example 6. Show that the area bounded by the closed curve $\boldsymbol{\gamma}(s) = (x(s), y(s))$, $s \in (a, b)$ (where $\boldsymbol{\gamma}(s)$ is not necessarily unit-speed) can be written as

$$A = \int_a^b x\dot{y}\,\mathrm{d}s,$$

Write down two other expressions for the area.

[2]For an interesting historical account of the isoperimetric and other optimization problems, I recommend *The Parsimonious Universe* by Hildebrant and Tromba, as well as this video lecture by John Barrow: `http://bit.ly/1Jwrz7g`

Lemma 3.5. $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$. Equality holds *iff*

Lemma 3.6. For all $A, B, C, D \in \mathbb{R}$,

$$(AB - CD)^2 \leq (A^2 + C^2)(B^2 + D^2),$$

where the equality holds *iff* $AD + BC = 0$.

Lemma 3.7. ["GM $\leq$ AM"] For all $A, B \geq 0$, we have

$$\sqrt{AB} \leq \frac{A+B}{2}.$$

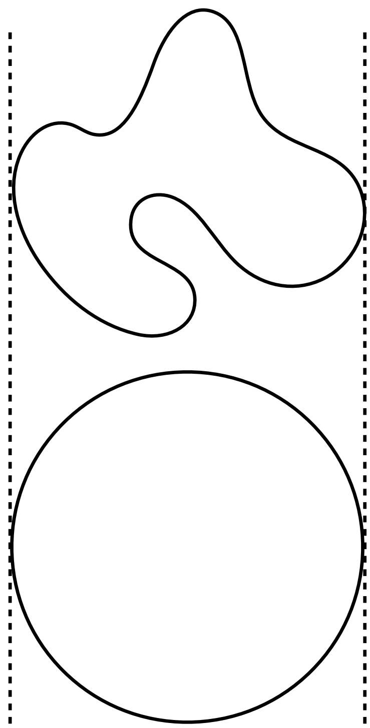
When does equality hold?

Theorem 3.8. [*Isoperimetric Inequality*] Let $\boldsymbol{\gamma}$ be a simple closed curve with length L, enclosed area A. Then

$$A \leq \frac{L^2}{4\pi},$$

and equality holds *iff* $\boldsymbol{\gamma}$ is a circle.

Proof. This proof is due to the German mathematician Erhard Schmidt (1876-1959).



Parametrize the given curve by arc-length:

$$\boldsymbol{\gamma}(s) =$$

Place the origin at the centre of a circle C (which does not intersect $\boldsymbol{\gamma}$) with diameter equal to the distance between the two vertical lines which meet and bound $\boldsymbol{\gamma}$ at $s = 0$ and $s = s_1$. Also assume that C and $\boldsymbol{\gamma}$ are both positively oriented.
Let the equation of the circle be

$$\mathbf{C}(s) = \qquad\qquad\qquad\qquad .$$

Note that there is no guarantee that the circle is unit-speed in this parametrization.
Also, we are free to choose ____________________.
Get a feel for this statement by tracing out the two curves on the left simultaneously with two fingers.

Using Green's Theorem (see Example 6), the area enclosed by $\boldsymbol{\gamma}$ (length L) is

$$A =$$

Similarly, the area enclosed by the circle (say, radius r),

$$\pi r^2 =$$

Adding the two areas gives:

$$A + \pi r^2 =$$

$$\heartsuit \qquad \leq$$

$$\diamondsuit \qquad \leq$$

Now apply the "GM≤ AM" inequality to the areas of $\boldsymbol{\gamma}$ and the circle.

This proves the Isoperimetric Inequality. It remains to show that equality holds *iff* $\boldsymbol{\gamma}$ is a circle. The "if" part is immediate. Let's suppose $A = L^2/4\pi$. We want to show that $(x(s), y(s))$ describes a circle.

Firstly, note that the "GM≤AM" becomes an equality *iff*:

The inequality $\diamondsuit$ (see Lemma 3.6) becomes an equality *iff*

But we also know that $x^2 + \bar{y}^2 = r^2$. Differentiating wrt s gives:

Since $\mathbf{C} = (\bar{x}, \bar{y})$ describes a circle, $\boldsymbol{\gamma}$ is also a circle (translated along the y direction).

Finally, it remains to show that if $\diamondsuit$ is an equality, then $\heartsuit$ is also an equality.

In summary, the Isoperimetric *Equality* holds *iff* $\boldsymbol{\gamma}$ is circle. $\square$

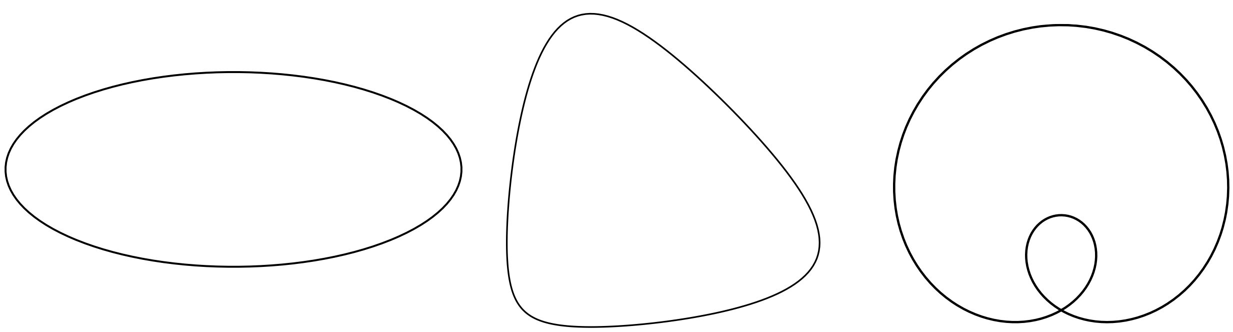
3.3 The Four-Vertex Theorem

Definition. A *vertex* of a plane curve $\boldsymbol{\gamma}(t)$ (not necessarily unit speed) is defined to be a point where the signed curvature attains a stationary point, *i.e.*

$$=$$

So a vertex is where the curve bends 'most' or 'least' *locally*. A circle has constant curvature, so every point on the circle can be regarded as a vertex. But think about an ellipse. How many vertices does it have?

Example 7. How many vertices do these figures have? Plot $\kappa_s(s)$ in each case.



The last Example shows a closed curve with _____ vertices. The curve is clearly not simple. Can you think of a *simple* closed curve with fewer than 4 vertices? The *Four-Vertex Theorem*, which we will prove, says that there is no such curve!

The Theorem actually concerns any simple closed curve, but in this course we will only prove a less general, though still useful, result for a subset of simple closed curves.

Definition. A closed curve $\boldsymbol{\gamma}$ is said to be ______________ if the line segment joining any two points on $\boldsymbol{\gamma}$ lies *entirely* in the interior[3] of $\boldsymbol{\gamma}$.

[3]It might seem *obvious* that that a simple closed curve divides the plane into two regions: one bounded (the interior), and the other unbounded (the exterior), but the proof is rather involved. This is the *Jordan Curve Theorem*, which you will meet in a topology course.

Example 8. Draw a convex and a non-convex closed curve.

Lemma 3.9. Let $\boldsymbol{\gamma}$ be a convex simple closed curve which is not a circle. If $\boldsymbol{\gamma}$ has up to 3 vertices, then there is a straight line dividing $\boldsymbol{\gamma}$ into two open segments: one on which $\dot{\kappa}_s > 0$ and on the other $\dot{\kappa}_s \leq 0$ (or $\dot{\kappa}_s \geq 0$ on one and $\dot{\kappa}_s < 0$ on the other).

Proof. : Let $\boldsymbol{\gamma}(s)$ be its arc-length parametrization and let L be its length. As $\boldsymbol{\gamma}$ is assumed to be smooth, $\kappa_s(s) = \ddot{\boldsymbol{\gamma}} \cdot \mathbf{n_s}$ is clearly a continuous function on the interval __________, and so it must attain a maximum and a minimum somewhere on that interval (____________________ theorem). Call these points $\mathbf{p}$ and $\mathbf{q}$ respectively (these points are vertices by definition). Consider the following 3 possibilities.

- If $\boldsymbol{\gamma}$ has only 1 vertex, we must have $\mathbf{p} = \mathbf{q}$ (max=min). Thus, κ_s is ____________ on the whole curve, *i.e.* it is a __________.

- If $\boldsymbol{\gamma}$ has only 2 distinct vertices, $\mathbf{p}$ and $\mathbf{q}$. Draw a straight line through $\mathbf{p}$ and $\mathbf{q}$. This line clearly separates $\boldsymbol{\gamma}$ into two segments, with κ_s increasing on one and decreasing on the other. Hence the Lemma holds in this case.

- If $\boldsymbol{\gamma}$ has 3 distinct vertices, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$. Clearly, κ_s on the three segments must be either increasing or decreasing. Since there must be two adjacent increasing (or decreasing) segments (with $\dot{\kappa}_s = 0$ at the joints), a similar line can be drawn as in the previous case, and the Lemma still holds. □

This strange Lemma simply says that any (non-circular) curve with up to 3 vertices must comprise two pieces on which $\dot{\kappa}_s$ has opposite signs. We will use this to prove...

Theorem 3.10. [*Four-Vertex Theorem: convex version*] Every convex simple closed curve in $\mathbb{R}^2$ which is not a circle has at least 4 vertices.

Proof. This elegant proof was given by S. Mukhopadhyaya in 1909.

Parametrize $\boldsymbol{\gamma}$ arc-length. Suppose $\boldsymbol{\gamma}(s)$ has up to 3 vertices. We will show that a contradiction follows, and therefore $\boldsymbol{\gamma}$ must have at least 4 vertices.

- By Lemma 3.9, there is a line ℓ which divides $\boldsymbol{\gamma}$ into two segments on which $\dot{\kappa}_s$ has opposite signs (with $\dot{\kappa}_s = 0$ only at the vertices).

- Let's set up the coordinate axes. Translate the curve along so that the line ℓ coincides with the x axis, with the origin falling within the curve (recall: translation and rotation don't change κ_s). Furthermore, we can place the segment with $\dot{\kappa}_s \geq 0$ on top (above the x axis). Let's draw the figure below.

- Let's focus on the y coordinate of $\boldsymbol{\gamma}(s) = (x(s), y(s))$.

  Recall that $\ddot{\boldsymbol{\gamma}} = \kappa_s \mathbf{n}_s$. Writing this in terms of $(x(s), y(s))$:

- Now from the setup shown in the figure, we see that the combination $\dot{\kappa}_s y$ is always non-negative. In fact, on the segment below the x axis, ________________.

  This implies that the integral

$$I = \int_0^L \dot{\kappa}_s \; y \; \mathrm{d}s \tag{3.6}$$

- However, integrating I by parts, and using the fact that x, y, κ (and their derivatives) are L-periodic, we find

This contradicts Eq. 3.6, which says that $I > 0$. $\square$

The convexity condition can be removed, although the proof (published by A. Kneser in 1912) takes a rather different approach using *stereographic projection* (which you will meet in problem sheet 4). We will simply state this general version here (you may state this version in the exam).

Theorem 3.11. [*Four-Vertex Theorem*] Every simple closed curve in $\mathbb{R}^2$ which is not a circle has at least 4 vertices.

Example 9. Consider the curve $\boldsymbol{\gamma}(t) = (p\cos t, q\sin t)$, where p and q are nonzero. From Example 5, its signed curvature is given by

$$\kappa_s(t) = \frac{pq}{[(p\sin t)^2 + (q\cos t)^2]^{3/2}}.$$

How many vertices does $\boldsymbol{\gamma}$ have?

Chapter 4

Surfaces

A curve looks almost like a line *locally*. A line is one-dimensional, so it is possible to describe a curve with one parameter.

Similarly, a surface is something that's looks almost like a __________ locally, and so requires _____ parameters to describe it. We will define it formally later.

The formal definition of surfaces is quite technical. Why? Well, as an analogy, we can easily understand what a *function* does, but it takes a bit of work to define them formally. With the formal definition, we were able to understand why, for instance, the circle is not a function, but is actually two functions that can be patched together.

Similarly, even though we think we know what looks like a surface, it takes a bit of work to define them formally, but when we get there we will understand why the sphere is (or is not?) a surface.

4.1 Prelude

Here are some ideas that we need in this Chapter. Again, you will have seen most of them before in analysis, calculus or vector calculus.

Definition. For $\mathbf{x} \in \mathbb{R}^n$, the *Euclidean norm* (or L^2 norm) in $\mathbb{R}^n$ is defined as

$$|\mathbf{x}| = \tag{4.1}$$

Make sure you use the right norm in the right dimension.

Definition. An *open ball*, $B_\varepsilon(\mathbf{x_0})$ in $\mathbb{R}^n$, is defined as the set

$$B_\varepsilon(\mathbf{x_0}) = \qquad (4.2)$$

where $\varepsilon > 0$ and $\mathbf{x_0} \in \mathbb{R}^n$.

The norm in this definition is the n-dimensional Euclidean norm. $B_\varepsilon(\mathbf{x_0})$ is also known as the ______________________ of $\mathbf{x_0}$.

Definition. The set $S \subseteq \mathbb{R}^n$ is said to be *open* if $\forall \mathbf{x} \in S, \exists \varepsilon > 0$ such that ___________.

Definition. A set is said to be *closed* if its _______________ is open.

Definition. A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\} \subset \mathbb{R}^n$ is said to be *linearly independent* if the equation

only has the trivial solution $c_1 = c_2 = \ldots = c_n = 0$.

Although linear independence is strictly a property of the *set* $\{\mathbf{v}_i\}$, a widespread abuse of terminology (but a rather convenient one) allows us to say that the *vectors* $\mathbf{v}_i$ themselves are linearly independent.

Definition. A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to be *continuous* at $\mathbf{x_0}$ if, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\Longrightarrow \qquad (4.3)$$

Remember: *a continuous function maps nearby points to nearby points.* Note that the norms in the equation above may be different.

Definition. A function $f : A \to B$ is said to be __________, or C^∞, at $\mathbf{x_0}$ if f has continuous partial derivatives of all orders on A.

Definition. The *grad* operator acting on $f(x, y, z)$ is defined as $\nabla f =$ _______________. (I tend to use the subscript notation for partial derivatives).

Theorem. The vector ∇f is _______________ to the surface $f(x, y, z) =$ constant. (For proof, see Chapter 6 of Vector Calculus.)

We will be looking at a map from one vector space to another. The thing that tells you whether such a map is well-behaved (invertible) is the ______________.

Definition. The *Jacobian matrix* for the map $f : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix

$$J = \frac{\partial f_i}{\partial x_j} = \tag{4.4}$$

When $n = m$, J is a square matrix. Its determinant, $|J|$ (sometimes also confusingly called "the Jacobian") determines whether the inverse function exists *locally*.

Example 1. Consider a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$F(u, v) = (\cos u \sin v, \sin u \sin v).$$

Find the Jacobian and the Jacobian determinant of this function.

Theorem. [*Inverse Function Theorem*] Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function. If the Jacobian $|J| \neq 0$ at a point $\mathbf{x_0} \in \mathbb{R}^n$, then there exists a neighbourhood U of $\mathbf{x_0}$ such that f^{-1} exists and is smooth (*i.e.* f is locally a __________________).

(compare the above with Lemma 1.6). In multi-variable calculus, the Jacobian plays an analogous role to the derivative in one-variable calculus.

The proof of the Inverse Function Theorem can be found in any analysis textbook.

Question. Is the function in Example (1) invertible?

Theorem. [*'Converse' to Inverse Function Theorem*] If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, then _________.

Here are some objects which are prototype of surfaces (yet to be defined).

Example 2. Let $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ be unit vectors in $\mathbb{R}^3$ that are linearly independent. Let Π be the plane which is parallel to these vectors, and passes through the point $\mathbf{a}$. Then Π can be written as the function $\boldsymbol{\sigma} : \mathbb{R}^2 \to \Pi$, with

$$\boldsymbol{\sigma}(u, v) = \qquad\qquad\qquad\qquad \text{where} \quad (u, v) \in \mathbb{R}^2. \tag{4.5}$$

Try not to think of the plane as a geometric object, but as the image of a map $\boldsymbol{\sigma}$ from $\mathbb{R}^2$ to $\mathbb{R}^3$.

Example 3. Assuming that $\hat{\mathbf{p}} \cdot \hat{\mathbf{q}} = 0$, find an explicit expression for $\boldsymbol{\sigma}^{-1}$ (the inverse function of $\boldsymbol{\sigma}$).

Example 4. A surface is described by the equation $z = \sqrt{x^2 + y^2}$. Explain why it has a rotational symmetry about the z axis. Sketch the surface.
Express the surface in the form $\boldsymbol{\sigma} : \mathbb{R}^2 \to \mathbb{R}^3$. Describe $\boldsymbol{\sigma}^{-1}$ in one word.

Example 5. Using spherical coordinates, write down the equation of the unit sphere in the form $\boldsymbol{\sigma} : U \to \mathbb{R}^3$, where $U = [0, \pi] \times [0, 2\pi)$. Is $\boldsymbol{\sigma}$ invertible?

Example 6. Using cylindrical coordinates, write down the equation of the unit cylinder in the form $\boldsymbol{\sigma} : U \to \mathbb{R}^3$, where $U = [0, 2\pi) \times \mathbb{R}$. Is $\boldsymbol{\sigma}^{-1}$ a continuous function?

As mentioned in the introduction, a surface can be defined as a 3D object which is *locally* 2-dimensional. The previous Examples inspire us to define a surface via a map from some subset of $\mathbb{R}^2$ to a subset of $\mathbb{R}^3$. We want this map to have 'nice' properties (*e.g.* continuous, invertible).

Definition. Two sets A and B are said to be ____________________ if there exists a function $f : A \to B$ such that

- f is continuous on A
- f is invertible
- $f^{-1} : B \to A$ is also continuous.

Such a function f is called a ____________________.

Intuitively, homeomorphic objects have 'essentially' the same shape. Think of a homeomorphism can as a sequence of continuous stretching and bending, but not tearing or gluing (*e.g.* a book is homeomorphic to a ball, a donut is homeomorphic to a tea cup, but a donut is not homeomorphic to a book.).

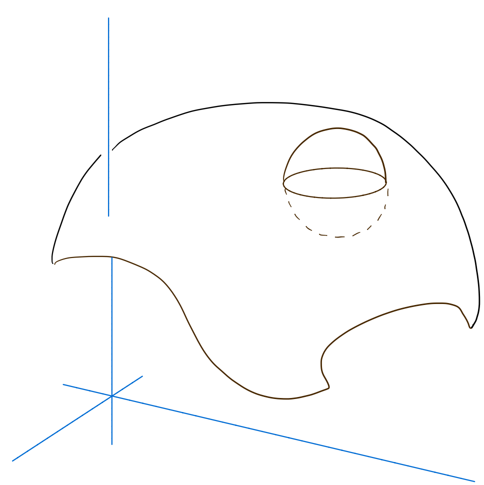
Question. What is the difference between a *homeomorphism* and a *diffeomorphism* defined in Chapter 1? Intuitively, which one is a 'nicer' function?

Example 7. Prove that the interval $(-1, 1)$ is homeomorphic to $\mathbb{R}$.

4.2 Definition of a surface

Intuitively, a surface is 'locally $\mathbb{R}^2$'. Here is how to formalise this idea.

Definition. A subset $S \subset \mathbb{R}^3$ is called a *surface* if, $\forall \mathbf{p} \in S$, there exists an $\underline{\text{open}}$ set $U \subseteq \mathbb{R}^2$, and an $\underline{\text{open}}$ set $V \subseteq \mathbb{R}^3$ containing $\mathbf{p}$, such that U is homeomorphic to $S \cap V$.



Definition. The homeomorphism $\boldsymbol{\sigma} : U \to S \cap V$ is called a ____________________ of the surface (or a *coordinate patch*, or a *chart*).

The reason we always work with open sets is to ensure that $\boldsymbol{\sigma}$ is invertible. This is also the reason why, in the previous chapters, we always tried to work with curves defined on open sets, so that $\boldsymbol{\gamma}$ may be regarded as a homeomorphism.

Note that a surface may have more than one patch/parametrization (as long as one exists at every point of S). This kind of 'patching' can also be done with curves.

Example 8. In this example, we will consider the unit circle as a surface (with one dimension suppressed). Study how the definition of a surface works in this case.
Show how the unit circle can be considered as a union of images of homeomorphisms defined on open sets.

Definition. The collection of all coordinate patches whose images cover the whole of S (the patches may overlap) is called an ____________ of S.

Example 9. Construct the atlas for the unit cylinder.

For the rest of this course, we will assume that any given $\boldsymbol{\sigma}(u, v)$ is a parametrization of a surface, *i.e.* $\boldsymbol{\sigma}$ is a homeomorphism, and that all the conditions in the definition of a surface are satisfied.

4.3 Regular surfaces

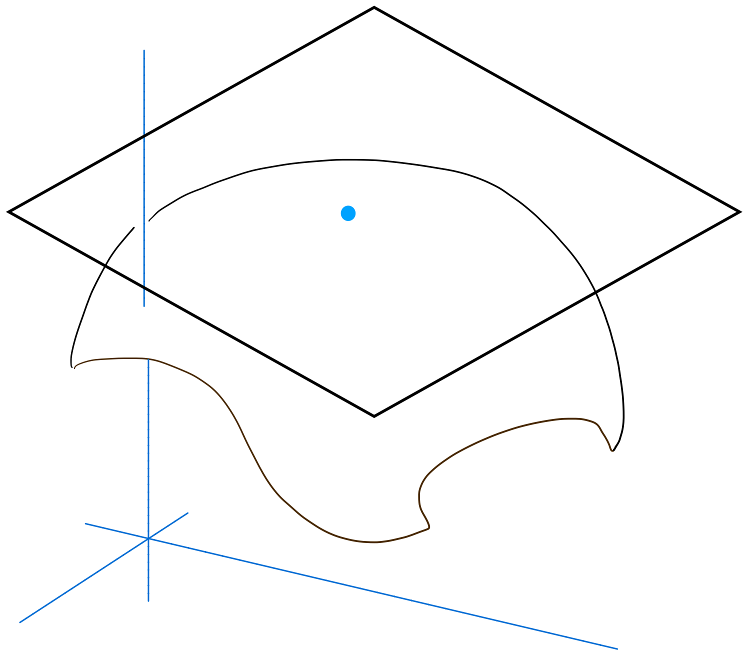
Recall why *regular* curves are important: since $\boldsymbol{\gamma}'(t) \neq \mathbf{0}$, a tangent vector is well defined at every point. Also, the entire curve can be swept out as t increases, never stopping and never backtracking (see sheet 1 Q4). We want a similar property for surfaces.

Unlike a curve, which has a unique tangent vector at a given point, there are clearly many possible tangent vectors at a given point on a surface.

Definition. A *tangent vector* to a surface S at $\mathbf{p} \in S$ is the tangent vector of any curve on S passing through $\mathbf{p}$.

Definition. The *tangent space* (or *tangent plane*), __________, is the set of *all* tangent vectors at $\mathbf{p}$.

Definition. A surface is said to be *regular* if there exists a parametrization $\boldsymbol{\sigma}(u, v)$ such that $\boldsymbol{\sigma}_u \equiv \frac{\partial \boldsymbol{\sigma}}{\partial u}$ and $\boldsymbol{\sigma}_v \equiv \frac{\partial \boldsymbol{\sigma}}{\partial v}$ are __________________________.
$\boldsymbol{\sigma}(u, v)$ is called a *regular parametrization* of the surface.



Let's try to make sense of this definition.

The vectors $\boldsymbol{\sigma}_u(\mathbf{p})$ and $\boldsymbol{\sigma}_v(\mathbf{p})$ are two tangent vectors at the point $\mathbf{p}$. If the surface is regular, then a tangent plane can be drawn at $\mathbf{p}$, with $\boldsymbol{\sigma}_u(\mathbf{p})$ and $\boldsymbol{\sigma}_v(\mathbf{p})$ lying on the plane. The linear independence simply ensures that the two tangent vectors span a plane.

In summary:

However, there is a crucial difference between regular curves and regular surfaces.

For curves, regularity is a property of a particular parametrization. You cannot look at the plot of a curve and tell whether it is regular or not.

For surfaces, regularity is a property of the surface itself (we only need an ideal $\boldsymbol{\sigma}(u, v)$ to exist). We can discuss whether a sphere or a plane is regular, without referring to any specific parametrization. Equally, given a bad parametrization $\boldsymbol{\sigma}(u, v)$, we cannot conclude that the surface is not regular.

Example 10. Sketch some surfaces that are not regular.

If $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ span the plane, then the vector ______________ is normal to the plane. We can use this observation to write down an equivalent definition of a regular surface.

Proposition 4.1. A surface is regular if and only if there exists a parametrization $\boldsymbol{\sigma}(u, v)$ such that the normal vector $\mathbf{N} = \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \neq \mathbf{0}$ everywhere on the surface.

Proof. This is an exercise in linear algebra (see sheet 4).

Example 11. Show that the following surfaces are regular.

a) The plane

b) The unit cylinder

Example 12. Consider a surface parametrized by $\boldsymbol{\sigma}(u, v) = (u \cos v, u \sin v, u)$, where $u > 0$ and $v \in (0, 2\pi)$. Is this a regular surface?

Example 13. Show that the surface parametrized by

$$\boldsymbol{\sigma}(u,v) = \left(\sqrt{1-u}\,\cos v, \sqrt{1-u}\,\sin v,\, u\right), \quad u < 1, v \in (0, 2\pi)$$

is regular. Find the equation of its tangent plane at $\mathbf{p}$ where $(u,v) = (0, \pi/4)$.
(We will sketch this surface in the next question.)

Now suppose that the surface has a closed-form expression $f(x, y, z) = 0$. Recall from vector calculus that the vector ________ is normal to the surface at any given point. By Prop. 4.1, we deduce the following result:

Theorem 4.2. Suppose a surface, S, is defined by the equation

$$f(x, y, z) = 0,$$

where f is smooth. If ________________ then S is a regular surface.

Caution! The converse does not hold! If $\nabla f = \mathbf{0}$, no conclusion about the regularity of the surface can be drawn (see sheet 4).

The proof is slightly technical and I won't go into here (See Pressley, page 118). We shall use this result without proof.

Example 14. Redo the previous example using Theorem 4.2. Sketch the surface.

Example 15. Use Theorem 4.2 to show that the following surfaces are regular.

(a) $x^2 + y^2 = 1$

(b) $x^2 + y^2 = z^2$, $z \neq 0$.

(c) $z = f(x, y)$, where f is a smooth function on $\mathbb{R}^2$.

4.4 Surface reparametrization

Just like curves, surfaces can be parametrized (patched) in many ways. Recall from Chapter 1 that when a regular curve is reparametrized, it is again regular. We will prove a similar result for surfaces here. But first, some revision on the Jacobian.

Example 16. Describe the surface parametrized by

$$\boldsymbol{\sigma}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)\,, \quad (u,v) \in (0,2\pi) \times (0,\pi/2).$$

(a) Show that the surface is regular.

(b) The surface can also be parametrized as $\tilde{\boldsymbol{\sigma}}(\tilde{u},\tilde{v}) = (\tilde{u},\tilde{v}, f(\tilde{u},\tilde{v}))$. Determine f.

(c) Write down a smooth map Φ such that $\Phi(u,v) = (\tilde{u},\tilde{v})$.

(d) Find the Jacobian determinant, $|J|$, for the transformation Φ. Is Φ invertible?

(e) Sketch a diagram (on the next page) illustrating the two parametrizations of this surface.

The definitions below are very similar to those we have studied for curves.

Definition. $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ is said to be a ______________________ of a surface $\boldsymbol{\sigma}(u, v)$ if there is a diffeomorphism Φ such that

$$\tag{4.6}$$

Definition. The diffeomorphism Φ linking two parametrizations of a surface is known as a *reparametrization* map or a _____________ map.

Proposition 4.3. Let $\boldsymbol{\sigma}(u, v)$ be a regular parametrization of a surface. Then, its reparametrization is also a regular parametrization of the surface.

Proof. Let the reparametrization map be $\Phi(\tilde{u}, \tilde{v}) = (u, v)$, where $u = u(\tilde{u}, \tilde{v})$ and $v = v(\tilde{u}, \tilde{v})$.

To check that $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ is a regular parametrization, we need to check that _______________.

Since $\boldsymbol{\sigma}$ is regular parametrization, we know that _______________. Also, since Φ is a diffeomorphism, _______________ (this is the converse of the Inverse Function Theorem).

Hence $\tilde{\boldsymbol{\sigma}}_{\tilde{u}} \times \tilde{\boldsymbol{\sigma}}_{\tilde{v}} \neq \mathbf{0}$, and so $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ is again a regular parametrization. □

Let's compare this with what happened in Chapter 1 for curves. We could talk about a *regular curve* rather than a "regular parametrization" of a curve because, once a curve is regular, its reparametrizations are again regular. If we find a dodgy parametrization of a regular curve which produces a non-regular curve, then that parametrization cannot be a 'reparametrization' (*i.e.* not a diffeomorphism).

As we have just shown, the results for regular surfaces are completely analogous.

Chapter 5

Surfaces II – Examples

5.1 Quadrics

The quadric surfaces are a 3D generalisation of the quadratic equation $ax^2 + bx + c = 0$.

Question. Guess the form of the quadric $f(x, y, z) = 0$.

Definition. A quadric is an equation of the form

$$\mathbf{x}^T A\mathbf{x} + \mathbf{b} \cdot \mathbf{x} + c = 0, \tag{5.1}$$

where $\mathbf{x} = (x\ \ y\ \ z)^T$, A is a symmetric 3×3 matrix, $\mathbf{b}$ is a constant vector, and $c \in \mathbb{R}$. One can easily multiply out the products to find that Eq. 5.1 is equivalent to the equation we guessed.

Example 1. Put the following in quadric form (5.1): a) the unit sphere, b) the double cone, c) The surface $x^2 + 2y^2 - 4z^2 + 2xy + yz - 6xz + 1 = 0$.

A general quadric with lots of cross terms can be reduced to a simpler form using a sequence of rotations and translations[1]. In fact, any quadric can be reduced one of the 14 *standard quadrics* listed on the next page.

You should find your own way of remembering their names. Clues to their shapes can be found by setting $x = 0$ or $y = 0$ or $z = 0$, and thinking about any apparent symmetries (e.g. with respect to rotation or reflection). We will study many interesting properties of quadrics in the upcoming chapters.

Here are some real-life quadrics. To which class do they belong?

[1]See Pressley page 101 for the proof based on the ideas outlined here.

Quadric	Equation ($a, b, c \neq 0$)
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Double elliptic cone	
	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
Elliptic cylinder	
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
	$y = \frac{x^2}{a^2}$
	$x = 0$
	$x^2 = a^2$
	$x^2/a^2 - y^2/b^2 = 0$
	$x = y = 0$
	$x = y = z = 0$

Table 5.1: The 14 standard quadrics.

5.2 Surfaces of revolution

A surface of revolution is generated by taking a curve $\boldsymbol{\gamma}(u)$ in $\mathbb{R}^3$ which is confined to the x-z plane, say

$$=$$

with $f(u) > 0$, and rotating it about the z axis.

We see that every horizontal slice of the surface is a circle of radius __________. This tells us about the x, y components of the 3D surface, which we can parametrize as

$$\tag{5.2}$$

Example 2. Give equations of two quadrics which are surfaces of revolution.
In each case, write down the equation of the generating curve $\boldsymbol{\gamma}(u)$ in the x-z plane, and the surface parametrization $\boldsymbol{\sigma}(u, v)$.

Lemma 5.1. A surface of revolution is a regular surface *if and only if* it is generated by a regular curve.

Definition. The *meridians* of a surface of revolution (5.2) are curves of constant _____. The *parallels* are the circles of constant _____.

If we think of the Earth as a sphere, then the meridians are just the _______________ and the parallels are the _______________.

Example 3. Prove that on a surface of revolution, parallels and meridians are orthogonal.

(more about parallels and meridians in Chapter 7.)

5.3 Ruled surfaces

Take a vertical line segment passing through $x = 1$ and rotate it around the z axis, the surface of revolution obtained is part of a ______________.

Take a line segment inclined at 45° in the x-z plane and rotate it around the z axis, the surface obtained is part of a ______________.

In fact, the segment could do all sorts of crazy acrobatics, but the surface it sweeps out is still a *union of straight lines*. This kind of surface is called a *ruled surface*.

Definition. A *ruled surface* can be parametrized by $\boldsymbol{\sigma}(u, v)$ which is *linear* in either u or v (or both). For example,

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\,\mathbf{a}(u), \tag{5.3}$$

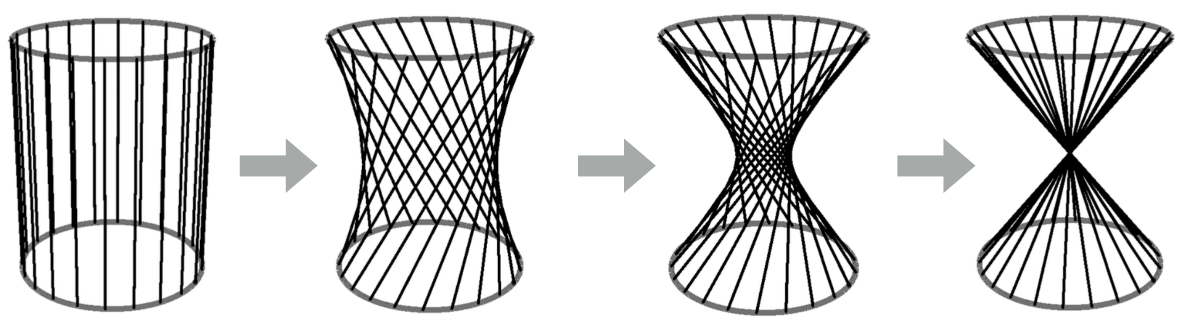
where $\boldsymbol{\gamma}$ and $\mathbf{a}$ are curves in $\mathbb{R}^3$.

Definition. The straight lines that generate the ruled surface are called ______________ (these are lines of constant _____ in (5.3)). The curve $\boldsymbol{\gamma}$ is called the ________________ (obtained by setting $v = 0$ in (5.3)).

Example 4. Show that the following are ruled surfaces. Which quadrics are they?
(a) The surface $x^2/a^2 + y^2/b^2 = 1$, (b) The surface $x^2 + y^2 = z^2$.

Example 5. Show that the surface parametrized by $\boldsymbol{\sigma}(u,v) = (\cos u - v\sin u,\ \sin u + v\cos u,\ v)$ is a ruled surface. Which quadric is this?
Identify the rulings and the base curve. Show that they always intersect at $45°$.

The picture below shows how the cylinder, hyperboloid (of 1 sheet) and cone can all be generated by twisting a pair of parallel rings joined by taut elastic strings[2].



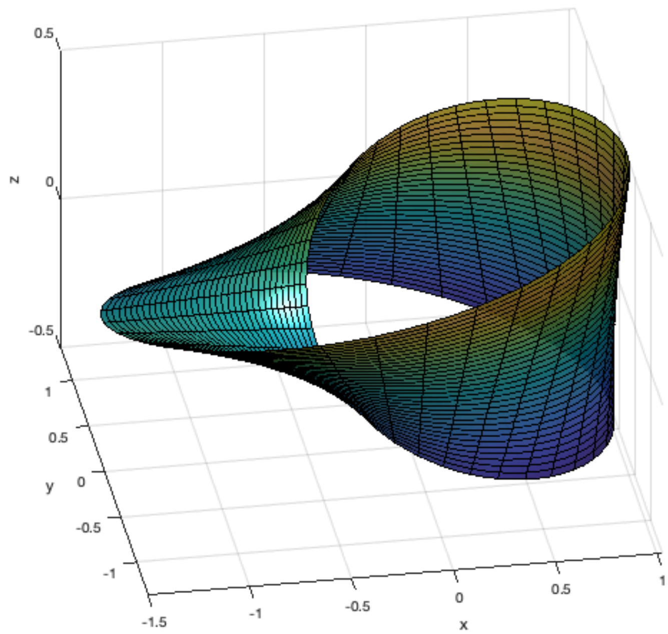
In fact, twisting the rings the other way would give the same surfaces, but with the rulings going in the opposite direction. This means that the hyperboloid of 1 sheet can be generated by two different families of straight lines, and is said to be ____________________. More about this in problem sheet 5.

(You can see this fact on the picture of the Corporation Street Bridge.)

[2]see a nice animation (with non-elastic strings) at `http://bit.ly/1iHcnio`

5.3.1 The Möbius strip

Let's finish this Chapter with a very special ruled surface, the Möbius strip[3]: Take a vertical line segment in the x-z plane, say, the line joining $(1,0,-1/2)$ to $(1,0,1/2)$, and rotate it around the z axis. But let's give the segment a $180°$ rotation before closing up.



It can be shown that this construction leads to the parametrization

$$\boldsymbol{\sigma}(u,v) = \left(\left(1 - u\sin\frac{v}{2}\right)\cos v,\ \left(1 - u\sin\frac{v}{2}\right)\sin v,\ u\cos\frac{v}{2}\right), \qquad (*)$$

where $u \in (-1/2, 1/2)$ and $v \in (0, 2\pi)$.

Question. Identify the base curve. Trace the path of the curve $u = 0.5$ in the above figure.

In problem sheet 5, you will show that the normal vector will also turn upside down as we go one complete round, *i.e.*, an 'outward' pointing normal will end up pointing 'inward'. Therefore, the Möbius strip has no well-defined notion of *inside* or *outside* (it only has one side).

A surface on which the normal vector field is discontinuous in u or v is called a ______________________________ surface. The Möbius strip is a classic example of such a surface.

[3] August Ferdinand Möbius (1790-1868), German mathematician. His other important contributions include *Möbius transformations* in complex analysis, and *Möbius function* in number theory.

Chapter 6

The First Fundamental Form

We will now study lengths, angles and areas on surfaces, all of which are *intrinsic* geometrical properties of surfaces, *i.e.* measurable by the surface inhabitants who do now know about the global properties of the surface (e.g. whether the surface has a hole).

The *first fundamental form* is the name of the quantity which contains *all* information on the intrinsic geometry of surfaces. The first fundamental form is one of the most important concepts in differential geometry.

6.1 Curves on surfaces

Take a curve $\boldsymbol{\gamma}(t)$ lying on a surface S parametrized by $\boldsymbol{\sigma}(u, v)$. Let $\mathbf{p}$ be a point on $\boldsymbol{\gamma}$.

The tangent __________ lies on the tangent plane at $\mathbf{p}$ (denoted __________). Recall that this tangent plane is spanned by the basis vectors __________ and __________. Therefore, for some $\lambda, \mu \in \mathbb{R}$, we can express $\boldsymbol{\gamma}'$ as the linear combination

(6.1)

On the other hand, points on the curve can also be expressed in terms of the surface. This means that we can treat u, v as functions of t and write

$$\boldsymbol{\gamma}(t) =$$

Differentiating wrt t and applying the Chain Rule, we find

$$\tag{6.2}$$

Comparing with (6.1) gives

$$\lambda = \qquad , \quad \mu = \qquad . \tag{6.3}$$

Now, recall from Chapter 1 that the arc-length, s, of $\boldsymbol{\gamma}(t)$ satisfies

$$\frac{\mathrm{d}s}{\mathrm{d}t} =$$

This can be expressed in terms of u, v as

$$\tag{6.4}$$

Lemma 6.1. Let $\boldsymbol{\sigma}(u(t), v(t))$ be the parametrization of a curve $\boldsymbol{\gamma}$ lying on a surface $\boldsymbol{\sigma}$. The arc-length of $\boldsymbol{\gamma}$ between $t = t_0$ and $t = t_1$ is given by

$$\tag{6.5}$$

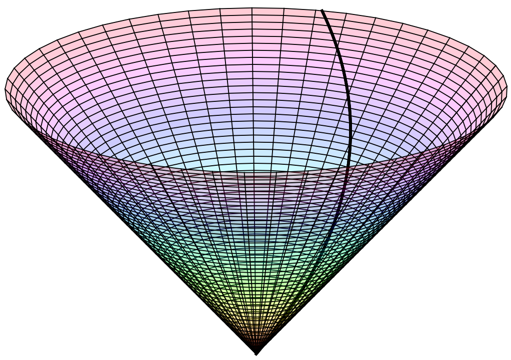
$$\text{where} \quad E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u, \quad F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v, \quad G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v$$

The above result is often expressed in a more traditional form without $\mathrm{d}t$.

Definition. The *first fundamental form* of the surface parametrized by $\boldsymbol{\sigma}(u, v)$ is

$$\mathrm{d}s^2 = \tag{6.6}$$

Example 1. Find the first fundamental form of the cone $\boldsymbol{\sigma}(u, v) = (u\cos v, u\sin v, u)$. Anton the ant lives on the cone. If he walks along the path $u = v = t$ from $t = 0$ to $t = 1$, find an integral expression for the distance he covered.



The first fundamental form helps us measure length on surfaces. This is why it's also known as the _______________ of the surface.

Example 2. Find the first fundamental form of the plane $\boldsymbol{\sigma}(u, v) = \mathbf{a} + u\hat{\mathbf{p}} + v\hat{\mathbf{q}}$, where $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ are orthonormal vectors.
Re-calculate the distance Anton covered if he were living on this plane instead.

Example 3. Find the first fundamental form of the surface of revolution

$$\boldsymbol{\sigma}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

Hence, obtain the first fundamental forms for the following surfaces:

(a) the unit sphere, expressed in the standard spherical coordinates,

(b) the surface $\boldsymbol{\sigma}(u, v) = (\cos v, \sin v, u)$.

Example 4. Prove that the coefficients of the first fundamental form always satisfy $EG - F^2 \geq 0$.

6.2 The derivative

The fact that the first fundamental forms of the cylinder and the plane are the same is rather curious. What exactly does this mean?

This motivates us to study what happens to points and curves when they are mapped from one surface onto another. This section is slightly technical but very important as it lays the foundation for the rest of the course. Make sure to study this section again.

Consider point $\mathbf{p}$ on a surface S. Recall that the tangent space $T_{\mathbf{p}}S$ is the union of the tangent vectors of *all* curves going through $\mathbf{p}$. Let $\boldsymbol{\gamma}(t)$ be any such curve, and WLOG let $\mathbf{p} = \boldsymbol{\gamma}(t = 0)$.

Take any vector $\mathbf{w}$ in $T_{\mathbf{p}}S$. Clearly $\mathbf{w}$ is a tangent to some curve $\boldsymbol{\gamma}$, so that $\mathbf{w} =$ __________.

We can also express $\mathbf{w}$ in terms of the basis of $T_{\mathbf{p}}S$ (Eq. 6.2) as ________________________ evaluated at $t = 0$.

Let f be a smooth map from S to $\tilde{S}$ (note that f maps 3D vectors to 3D vectors). This means, for instance, that $\mathbf{p} \in S$ is mapped to __________ $\in \tilde{S}$, and a curve $\boldsymbol{\gamma}(t)$ on S is mapped to ______________________________ on $\tilde{S}$. Note that we are using the same parameter t to parametrize both the curve on S and its image on $\tilde{S}$ (see diagram below).

The tangent vector $\tilde{\mathbf{w}}$ to the image curve is given by

$$\tilde{\mathbf{w}} = \tag{6.7}$$

This vector lies in the tangent space _____________. The map which takes $\mathbf{w}$ to $\tilde{\mathbf{w}}$ (mapping one tangent space to another) is called the _______________, simply denoted as __________ (since the notation in (6.7) is quite cumbersome).

Definition. Given a smooth map $f : S \to \tilde{S}$ and $\mathbf{p} \in S$, the *derivative* $D_{\mathbf{p}}f$ is defined as

$$D_{\mathbf{p}}f : \qquad\qquad\qquad \text{such that} \quad D_{\mathbf{p}}f(\mathbf{w}) = \tilde{\mathbf{w}}. \tag{6.8}$$

where $\tilde{\mathbf{w}}$ is given by (6.7).

In short, f maps a surface to a surface, but $D_{\mathbf{p}}f$ maps a tangent space to a tangent space. Note that the derivative depends on the map f and the base point $\mathbf{p}$. It's helpful to read $D_{\mathbf{p}}f$ as:

Equation 6.7 looks as though the the derivative also depends on the choice of the curve $\boldsymbol{\gamma}$ through $\mathbf{p}$, but actually this is not the case, as we will not prove. In addition, the derivative takes on a simple form when written as a matrix.

Lemma 6.2. The derivative, $D_{\mathbf{p}}f$, is a *linear* map which can be represented by the matrix multiplication

$$D_{\mathbf{p}}f(\mathbf{w}) = \qquad\qquad\qquad . \tag{6.9}$$

In particular, the derivative is independent of the choice of the curve $\boldsymbol{\gamma}$ through $\mathbf{p}$.

Proof. The setup is the same as before (refer to previous diagram).

Let $\boldsymbol{\sigma}(u, v)$ be the parametrization of S and let $\mathbf{p} =$ __________ for some curve $\boldsymbol{\gamma}$.

Since $\mathbf{p}$ lies on the surface, we can also write $\mathbf{p} =$ ____________________.

Under the map $f : S \to \tilde{S}$, let $f \circ \boldsymbol{\sigma}(u, v) =$ _____________, where $\tilde{u} = \tilde{u}(u, v)$ and

Now take $\mathbf{w} \in T_{\mathbf{p}}S$. Let's try to express the image, $\tilde{\mathbf{w}}$, in terms of the basis of the new tangent space, namely, __________ and __________.

This is all quite messy, but actually things look much simpler if we recast our findings in terms of *matrices* written in different bases.

Since $\mathbf{w} \in T_{\mathbf{p}}S$, we can express it in terms of the basis of $T_{\mathbf{p}}S$ as

$$\mathbf{w} = \lambda \boldsymbol{\sigma}_u + \mu \boldsymbol{\sigma}_v$$

where $\lambda =$ __________ and $\mu =$ __________ (see Eqs. 6.1 and 6.2). Expressing this as a vector written in terms of the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$:

$$\mathbf{w} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}.$$

Similarly, we can express $\tilde{\mathbf{w}}$ in the basis $\{\tilde{\boldsymbol{\sigma}}_{\tilde{u}}, \tilde{\boldsymbol{\sigma}}_{\tilde{v}}\}$. Indeed, our long calculations give:

where J is the usual Jacobian matrix for the transformation $(u, v) \to (\tilde{u}, \tilde{v})$. In this context, J acts as the ____________________ matrix between bases $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ and $\{\tilde{\boldsymbol{\sigma}}_{\tilde{u}}, \tilde{\boldsymbol{\sigma}}_{\tilde{v}}\}$.

The matrix equation shows that the derivative is independent of the choice of γ. $\square$

6.3 Induced dot product

Imagine we know all about lengths and angles on one surface S. How are lengths and angles measured on $f(S) = \tilde{S}$? We now show that the derivative gives us the necessary tool to perform these measurements on $\tilde{S}$.

Given our study of the derivative, we can write down how f maps a dot (or *inner*) product on $T_{\mathbf{p}}S$ to a dot product on $T_{f(\mathbf{p})}\tilde{S}$.

$$\langle \mathbf{v}, \mathbf{w} \rangle \equiv \mathbf{v} \cdot \mathbf{w} \rightarrow \tag{6.10}$$

Each side of Eq. (6.10) is a function of any two tangent vectors in $T_{\mathbf{p}}S$. We can express Eq. (6.10) more abstractly in *operator* form (without the vector arguments) as:

$$\tag{6.11}$$

where the subscript f denotes the fact that this dot product is __________ by f. Each of these functions takes a pair of vectors and spits out a number.

We can similarly define the *induced norm* $|\cdot|_f$ as follows. Denote the norm $|\mathbf{w}| \equiv \sqrt{\langle \mathbf{w}, \mathbf{w} \rangle}$. Under the map f, the *induced norm* is defined as:

Lemma 6.3. For any vector $\mathbf{v}$ and $\mathbf{w}$,

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{2}\left(|\mathbf{v} + \mathbf{w}|^2 - |\mathbf{v}|^2 - |\mathbf{w}|^2\right)$$

6.4 Local isometry

We now show that if a map f *preserves* the dot product, then this is equivalent to the fact that f preserves lengths and angles on surfaces.

Definition. Let S and $\tilde{S}$ be surfaces. The map $f : S \to \tilde{S}$ is called a ____________________ if, at every point $\mathbf{p} \in S$,

for all $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}S$.

This is what we mean when we say that "the dot product is *preserved*" by a local isometry from S and $\tilde{S}$. We can also say that S and $\tilde{S}$ are _________________________.

Theorem 6.4. The map $f : S \to \tilde{S}$ is a local isometry *iff* it preserves the first fundamental form.

Proof. Let S be parametrized by $\boldsymbol{\sigma}(u, v)$. For example, note that on S, the coefficient $E =$____________, whereas on $\tilde{S}$, $\tilde{E} =$ ____________.

If f is a local isometry. We want to show that S and $\tilde{S}$ have the same first fundamental form. Since f preserves the dot product, we see that:

Thus, f preserves the first fundamental form.

Conversely, if f preserves the first fundamental form, we want to show that

Take any $\mathbf{w} \in T_{\mathbf{p}}S$. We can write $\mathbf{w}$ as a linear combination of _____ and _____ as:

This means that $|\cdot| = |\cdot|_f$, *i.e.* f preserves length of a vector. By Lemma 6.3, f also preserves the dot product. Therefore, f is a local isometry. $\square$

Recall from the beginning of this chapter that the first fundamental form determines the length of curves on surfaces. Hence, we immediately have the following results.

Corollary. The following are equivalent.

- f is a local isometry.
- f preserves the ________________________________.
- f preserves the ______________________ on $T_{\mathbf{p}}S$ and on $T_{f(\mathbf{p})}\tilde{S}$.
- f preserves the ______________________ on S and on $\tilde{S}$.

These results give us several ways to check whether two surfaces are locally isometric. It's usually easiest to work with the first fundamental form.

Corollary. The cylinder (Example 3) and the plane (Example 2) are locally isometric.

Example 5. Show that the surface $\boldsymbol{\sigma}(u,v) = (u\cos v, u\sin v, u)$ with $0 < u < 1$ and $0 < v < 2\pi$, is locally isometric to the surface

$$\tilde{\boldsymbol{\sigma}}(\tilde{u},\tilde{v}) = \left(\sqrt{2}\tilde{u}\cos\frac{\tilde{v}}{\sqrt{2}},\ \sqrt{2}\tilde{u}\sin\frac{\tilde{v}}{\sqrt{2}},\ 0\right), \quad \text{with } 0 < \tilde{u} < 1,\ 0 < \tilde{v} < 2\pi.$$

6.4.1 The meaning of *local*

The previous results give us an intuitive meaning behind the word *isometry* (from Greek, *isos*= equal, *metria*= measure). If two surfaces are isometric, then the inhabitants on both surfaces see the same lengths and angles. But what exactly is *local* about this?

The plane and the cylinder are locally isometric. Intuitively, this means that, a small piece of a cylinder looks like a plane to its inhabitants.

But globally there is a difference – you can't smoothly transform a cylinder to a plane (a pair of scissors is needed).

Definition. A global *isometry* is a local isometry which is also a homeomorphism.

If S and $\tilde{S}$ are globally isometric surfaces, all continuous paths between two points on S are mapped to those on $\tilde{S}$ (and vice versa) in a 1-to-1 correspondence, and all these corresponding paths have the same length (thanks to the length-preservation property). In particular, this means that the *shortest* path between two given points on S is also mapped to the shortest path on $\tilde{S}$.

We can now see heuristically why the cylinder can't be *globally* isometric to the plane: The shortest distance between two points is not necessarily preserved.

In fact, the cylinder and the plane aren't even homeomorphic. To see this intuitively, note that a simple closed curve in the plane can be shrunk to a point (without leaving the plane. This property should hold under a homeomorphism (*i.e.* stretching and bending). However, this property does not necessarily hold on the cylinder.

However, if the two points are within a sufficiently small neighbourhood, then there's no problem: shortest distances are preserved, and closed curves can shrink to a point. This is the sense in which the isometry between these surfaces is only a *local* one.[1]

[1] We can make these statements more rigorous by invoking the inverse function theorem: a smooth map from S to $\tilde{S}$ is a homeomorphism in some neighbourhood of $\mathbf{p} \in S$ if the Jacobian of the mapping is nonzero. Thus, a local isometry is a global isometry within a sufficiently small neighbourhood.

6.4.2 What does all this mean for Anton the ant?

Anton the ant lives on an A4-paper world. He has studied the geometry of his world by measuring lengths of between places, angles between roads, area of his local park *etc.* and has written a book compiling all his measurements. He is unaware of, and unable to explore, the 3D space outside the surface.

One day, strong wind blows the piece of paper on which Anton lives, landing it in a strange deformed configuration (luckily it's not creased, torn or somehow perfectly rolled and glued into a tube).

The wind has now died down, and Anton then sets about remeasuring all the results that he compiled in his book. What would he now find?

He will find that *all his measurements are still exactly the same.* In this Chapter, we have shown that his measurements of lengths will be unchanged. In Chapter 7, we will show that his measurements of *angles* and *areas* are also unchanged.

A surface inhabitant like Anton who can only measure *intrinsic* geometry of the surface (*e.g.* length, angle, area) is unable to decide whether he lives on a plane or any of the shapes shown below, which are all locally isometric to the plane. This is true despite the fact that to an observer outside the surface, it is easy to distinguish between them.

In fact, Anton is even able to measure a kind of *curvature* of his new world, called the *Gaussian curvature*, which is also unchanged everywhere despite the deformation of the paper. This is the famous *Theorema Egregium* ('remarkable theorem') of Gauss. More about this in Chapter 8.

Chapter 7

Conformal and Equiareal maps

A (local) isometry preserves length on different surfaces. In this Chapter, we will study special maps that preserve angle or area on surfaces.

7.1 Angle-preserving maps

Suppose that we have two curves $\boldsymbol{\gamma}_1(t)$ and $\boldsymbol{\gamma}_2(t)$ on surface S parametrized by $\boldsymbol{\sigma}(u,v)$. If the curves intersect at $\mathbf{p} = \boldsymbol{\gamma}_1(0) = \boldsymbol{\gamma}_2(0)$ (we can always rescale t so that this happens), what is the angle at which the curves intersect?

The answer is simply the angle, θ, between the *tangent vectors* to the curves at $\mathbf{p}$. Thus, calculating the angle between curves reduces to calculating the angle between vectors on $T_{\mathbf{p}}S$. But writing these tangent vectors (say, $\mathbf{v}$ and $\mathbf{w}$) as linear combinations of _____ and _____, we see that

and thus the problem further reduces to finding the angle between those basis vectors, *i.e.* the first fundamental form.

Lemma 7.1. Let S be parametrized by $\boldsymbol{\sigma}(u,v)$ and $\mathbf{p} \in S$. The angle, θ, between $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ on $T_{\mathbf{p}}S$ can be expressed in terms of the first-fundamental-form coefficients as

$$\cos\theta =$$

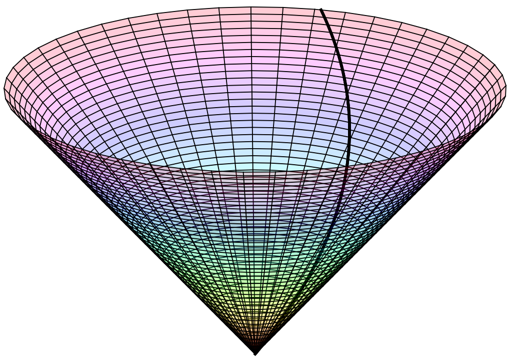
Proof:

Example 1 (An extension of Example 5 in Chapter 5)**.** Find an expression for the angle between $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ on the hyperboloid of 1-sheet

$$\boldsymbol{\sigma}(u,v) = (\cos u - v\sin u,\ \ \sin u + v\cos u, v).$$

Indicate this angle on a sketch. Explain what happens to the angle as v increases from 0 to infinity.

Example 2. Recall when Anton the ant walked on the cone in Chapter 6 – he walked along the path $u = v = t$ from $t = 0$ to $t = 1$. Calculate the angle that his trajectory makes with the rim of the cone.



Let's now consider a map, $f : S \to \tilde{S}$, between 2 surfaces. The curves $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ (intersecting at angle θ on S) are mapped onto two intersecting curves __________ and __________. The new angle of intersection could potentially be different from θ.

Again this problem reduces to comparing two angles:

- the angle, θ, between vectors $\mathbf{v}$ and $\mathbf{w}$ on $T_{\mathbf{p}}S$
- the angle, $\tilde{\theta}$, between

Definition. A diffeomorphism[1] $f : S \to \tilde{S}$ is said to be a ________________________ if $\theta = \tilde{\theta}$ (as defined above). In other words, f is conformal if, for all $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}S$,

$$= \tag{7.1}$$

It's easy to remember this definition in words:

Theorem 7.2. The diffeomorphism $f : S \to \tilde{S}$ is a conformal map *iff* there exists a function $\lambda : S \to \mathbb{R}$, with $\lambda(\mathbf{p}) > 0$ for all $\mathbf{p} \in S$, such that

$$\tag{$\star$}$$

for all $\mathbf{p} \in S$ and $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}S$.

[1] we require f to be a diffeomorphism so that regular curves are mapped onto regular curves, *i.e.* their tangents don't become degenerate. See exercise 4.4.4 in Pressley for details.

Proof. Suppose ($\star$) holds.

To prove the converse, see problem sheet 7 for a step-by-step guide. □

By letting $\mathbf{v}$ and $\mathbf{w}$ be $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ in ($\star$), we see that:

Corollary. f is a conformal map *iff* the first fundamental forms of S and $\tilde{S}$ are related by

Example 3. Prove that all local isometries are conformal maps.

Example 4. Using standard identities for cosh and sinh, prove the following identities:

$$\operatorname{sech}^2 u + \tanh^2 u = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}u} \operatorname{sech} u = -\operatorname{sech} u \tanh u, \qquad \frac{\mathrm{d}}{\mathrm{d}u} \tanh u = \operatorname{sech}^2 u.$$

Show that the surface parametrized by

$$\boldsymbol{\sigma}(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u) \tag{7.2}$$

is conformal to a plane. What surface is this?

Definition. A surface S is said to be ________________________ if its first fundamental form satisfies

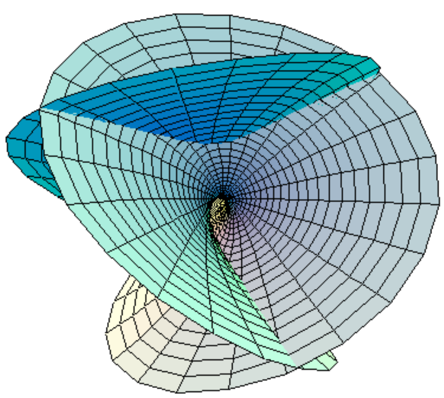
$$\mathrm{d}s^2 = \lambda^2(u,v)(\,\mathrm{d}u^2 + \,\mathrm{d}v^2).$$

Conformally flat surfaces are particularly interesting because angle measurements alone (*i.e.* trajectories of light rays, trajectories of ships sailing on the surface...) will not be able to distinguish the surface from a plane.

Example 5. *Enneper's surface* is parametrized by

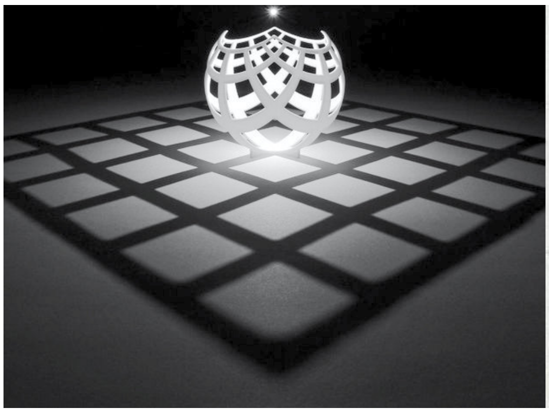
$$\boldsymbol{\sigma}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right).$$

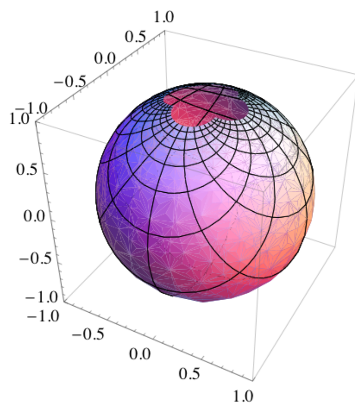
Show that $E = (1 + u^2 + v^2)^2$. Is it conformally flat?

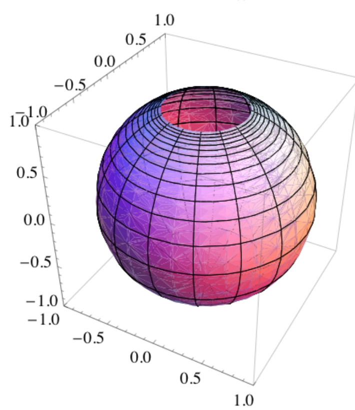


Here is another important conformal map. the stereographic projection from sheet 4. In sheet 7 you will prove that it is a conformal map.

The pictures below show some variants of this projection[2]. Note that perpendicular lines on the 2D grid correspond to perpendicular lines on the sphere, but the area of each grid element is distorted.







[2] *Left:* See Henry Segerman's website `http://www.segerman.org` for more 3D printed maths objects. *Right:* Rubens (1577-1640) was commissioned to draw this for the cover of a textbook on optics.

7.2 Area-preserving maps

Consider a surface parametrized by $\boldsymbol{\sigma}(u, v)$. What is the area on the surface over some given range of u and v?

Let's divide the area into small parallelograms as shown, and consider the vectors representing the two sides of the parallelogram.

Proposition 7.3. The area of the portion of the surface parametrized by $\boldsymbol{\sigma}(u, v)$ corresponding to the domain where $(u, v) \in R$ is given by

(7.3)

Proposition 7.4. The area of a parametrized surface is unchanged by a reparametrization. This echoes the result for the length of a curve which is unchanged by a reparametrization. I have included the proof of this statement at the end of this Chapter.

Example 6. Using spherical coordinates, calculate the surface area of the unit sphere.

The area formula (7.3) could be used as it is, but it's often easier to express it in terms of the first-fundamental-form coefficients. The resulting formula has no cross products.

Definition. The *metric tensor* of a parametrized surface is defined as the matrix

$$g \equiv \tag{7.4}$$

Here is a handy identity to help you avoid calculating a cross product. This is useful not only when dealing with area of surfaces, but also for showing that a surface is regular.

Proposition 7.5. $|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v| =$ __________

Proof. Recall the following formula from vector calculus

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}),$$

(you should try proving this using the ε_{ijk} technique). Now substitute ...

Lemma 7.6. The area of the surface $\boldsymbol{\sigma}(u, v)$ with metric g is given by

(7.5)

Example 7. Recalculate the surface area of the sphere using the first fundamental form.

Definition. A map $f : S \to \tilde{S}$ is said to be ______________ if it maps any region in S to a region of the same area in $\tilde{S}$.

We saw in Theorem 6.4 that a map is length-preserving *iff* it preserves g. The next theorem shows that a map is *area*-preserving *iff* it it preserves __________.

Theorem 7.7. Let $f : S \to \tilde{S}$ be a diffeomorphism, with S and $\tilde{S}$ parametrized by $\boldsymbol{\sigma}(u, v)$ and $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$. Then, f is equiareal *iff*

(7.6)

where $E = \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_u \rangle$ and $\tilde{E} = \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_u \rangle_f$, *etc.* .

Proof. If (7.6) holds, then the area is clearly preserved. Conversely, if

$$\iint_R \sqrt{\det g}\ \mathrm{d}u\,\mathrm{d}v = \iint_R \sqrt{\det \tilde{g}}\ \mathrm{d}u\,\mathrm{d}v$$

for all regions R in the domain, then we must have $\det g = \det \tilde{g}$. See problem sheet 7 for details on this last step. $\square$

7.2.1 Cylindrical projection

Have you ever noticed that the surface area of a sphere equals that of the cylinder which exactly contains it?

We now study one of the most famous equiareal maps, attributed to Archimedes who discovered this in $\sim$ 250BC.

Archimedes showed, using rudimentary geometrical arguments (remember this is pre-calculus era), that any horizontal strip on the sphere has the same area as the same strip on the cylinder which exactly contains the sphere.

Archimedes' tombstone (now lost) was said to be inscribed with a sphere contained within a cylinder. Here we study a proof of his result using differential geometry.

Consider the *cylindrical projection*, f, mapping the sphere to the cylinder where

$$\begin{aligned}\boldsymbol{\sigma}(u,v) &= (\sqrt{1-u^2}\cos v, \sqrt{1-u^2}\sin v, u), \quad u \in \\ \tilde{\boldsymbol{\sigma}}(u,v) &= (\cos v, \sin v, u), \quad u \in\end{aligned}$$

(I've dropped the pesky tildes). Note that the z coordinates of the sphere and cylinder are the same.

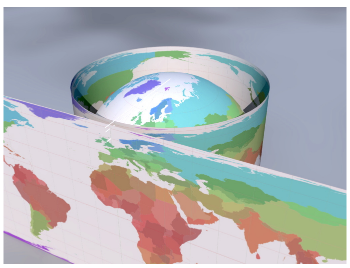
Theorem 7.8. (*Archimedes' Theorem*) The cylindrical projection is equiareal.
Proof:

Thus, we can choose any region, R, on the sphere and project it horizontally to a region, $\tilde{R}$, on the wrapping cylinder. R and $\tilde{R}$ will have the same area.

In particular, the patches we chose cover the entire sphere and cylinder (with straight lines removed – they have zero area anyway), so the entire sphere and the cylinder have the same surface area.

Question. Is the cylindrical projection a conformal map? Is it a local isometry?

The cylindrical projection can be used in *cartography* (map-making). The resulting map, called the *Lambert cylindrical projection* of the Earth, is one which does not distort areas (unlike the popular *Mercator* projection which you will study in problem sheet 7). However, the Lambert projection distorts lengths and angles.



Question. Is there a map of the Earth that preserves both area and angle?

We'll come back to this question in the next (and final) Chapter.

7.3 Summary

Type of map	Defining property	What's preserved?

Optional extra:

Proof of Proposition 7.4: The area of a parametrized surface is unchanged by a reparametrization.

Proof. Let the surface S be parametrized by $\boldsymbol{\sigma}(u, v)$ with $(u, v) \in U$. Suppose it can also be parametrized as $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ with $(\tilde{u}, \tilde{v}) \in \tilde{U}$. Let the surface areas under the two parametrizations by A and $\tilde{A}$.

From the proof of Proposition 4.3, we showed that

$$\tilde{\boldsymbol{\sigma}}_{\tilde{u}} \times \tilde{\boldsymbol{\sigma}}_{\tilde{v}} = \det J \, (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v).$$

where J is the Jacobian of the reparametrization $\Phi(\tilde{u}, \tilde{v}) = (u, v)$. Therefore,

$$\begin{aligned} \tilde{A} &= \int_{\tilde{R}} |\tilde{\boldsymbol{\sigma}}_{\tilde{v}} \times \tilde{\boldsymbol{\sigma}}_{\tilde{v}}| \, \mathrm{d}\tilde{u} \, \mathrm{d}\tilde{v} \\ &= \int_{\tilde{R}} |\boldsymbol{\sigma}_v \times \boldsymbol{\sigma}_v| |\det J| \, \mathrm{d}\tilde{u} \, \mathrm{d}\tilde{v} \\ &= \int_{R} |\boldsymbol{\sigma}_v \times \boldsymbol{\sigma}_v| \, \mathrm{d}u \, \mathrm{d}v \qquad (\dagger) \\ &= A, \end{aligned}$$

where at step (†), we used the formula for a change of area element in 2D integrals (see Vector Calculus). □

Alternatively, we can work with the metric determinant instead of the cross product. Using question 3 of sheet 4, we showed that under the reparametrization Φ, the metric g transforms as

$$\tilde{g} = J^T g J \implies \det \tilde{g} = (\det J)^2 \det g.$$

Square-rooting and integrating both sides with respect to $\tilde{u}$ and $\tilde{v}$ then lead to the same conclusion.

Chapter 8

Curvature of surfaces

8.1 Motivation

How do we measure the curvature of a surface? Recall that for a plane curve, $\boldsymbol{\gamma}(t)$, its curvature is completely characterised by the signed curvature, κ_s, which, according to problem sheet 2, satisfies the equation

$$\dot{\mathbf{n}}_s = -\kappa_s \mathbf{t} \implies$$

where $\mathbf{t}$ is the tangent vector and $\mathbf{n}_s$ is the ___________________ (see §2.2). Loosely, we can say that for a plane curve, its signed curvature is the (negative of the) rate of change of the normal vector ______________ in the direction of the tangent vector. It makes intuitive sense that a curve whose normal changes rapidly seems to curve *more* than one whose normal does not change much.

We could try to extend this idea to quantify how much a surface curves. Recall from sheet 5 that the standard unit normal of a surface $\boldsymbol{\sigma}(u, v)$ is defined as

$$\hat{\mathbf{N}} \equiv$$

The "rate of change" of this quantity is captured by two partial derivatives: _______ and _______. We then project (or dot) them in the direction of some unit tangent vector, which, for surfaces, is a linear combination of _______ and _______.

Thus, the quantities that could help us measure the curvature of a surface are:

(8.1)

(actually we will show later that two of them are equal). We will use these quantities to define the *second fundamental form*. We will show that the first *and* second fundamental forms together completely capture the information about the curvature of a surface.

8.2 The Gauss map

Let's examine the unit normal $\hat{\mathbf{N}}$ more carefully.

Any unit vector can be thought of as a vector joining the origin to a point on the unit sphere, and $\hat{\mathbf{N}}$ is no exception. In this sense, we can think of $\hat{\mathbf{N}}$ is a *map* from a point $\mathbf{p}$ on a surface S, to a point $\hat{\mathbf{N}}(\mathbf{p})$ on the unit sphere.

We will be talking about the unit sphere a lot so let's refer to it by the standard nickname[1] $\mathbb{S}^2$, *i.e.*

(8.2)

Definition. The *Gauss map*, $\hat{\mathbf{N}} : S \to \mathbb{S}^2$, is defined as follows: $\forall \mathbf{p} \in S$, $\hat{\mathbf{N}}(\mathbf{p})$ is the standard unit normal at $\mathbf{p}$.

Thus, the Gauss map is simply a fancy way of thinking about the standard unit vector as a mapping (rather than a vector).

[1] $\mathbb{S}^1$ is the unit circle. $\mathbb{S}^n$ is defined by the equation $x_1^2 + x_2^2 + \ldots x_{n+1}^2 = 1$

Example 1. No calculations allowed. Describe the image of the Gauss map of
a) The unit sphere, b) the unit cylinder, c) a plane.

In the above examples, we have assumed that the normals are all pointing upwards or outwards. What would happen if we assume the opposite orientation of the normals in the above examples?
Note that the concepts of in/out/up/down do not make sense on non-orientable surfaces like the ____________________. In this Chapter we will only deal with orientable surfaces.

Question. (A challenge) Describe the image of the Gauss map of the Möbius strip.

Example 2. Calculate the Gauss map $\hat{\mathbf{N}}$ of the surface $\boldsymbol{\sigma}(u,v) = (u\cos v, u\sin v, u^2)$, where $u > 0$ and $v \in (0, 2\pi)$. Calculate $\hat{\mathbf{N}}$ and describe the image of $\hat{\mathbf{N}}$.

It's worth clarifying our notation at this point. $\hat{\mathbf{N}}$ can be regarded as either a *vector*, or a *map*. We will not distinguish our notation as it should be clear from the context in what form we are regarding $\hat{\mathbf{N}}$ as. In particular, note that in this equation:

$$\hat{\mathbf{N}} = \hat{\mathbf{N}} \circ \boldsymbol{\sigma}(u, v), \tag{8.3}$$

$\hat{\mathbf{N}}$ on the LHS is a vector, but on the RHS it is regarded as a map.

Before we proceed further, let's state a couple of nice observations (easy to prove). These are analogous to the Frenet frame for curves.

Lemma 8.1. Let $\boldsymbol{\sigma}(u, v)$ be a regular surface with first fundamental form coefficients E, F, G.

- $\{\hat{\mathbf{N}}, \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ form a __________ of $\mathbb{R}^3$.
- If $F = 0$, the basis is ____________________.
- If $E = G = 1$ and $F = 0$, then the basis is ____________________.

Example 3. Prove that

(a) $\hat{\mathbf{N}}_u \cdot \hat{\mathbf{N}} = \hat{\mathbf{N}}_v \cdot \hat{\mathbf{N}} = 0$.

(b) $\hat{\mathbf{N}}_u \cdot \boldsymbol{\sigma}_u = -\hat{\mathbf{N}} \cdot \boldsymbol{\sigma}_{uu}$ and find 3 other similar results.

Deduce that $\hat{\mathbf{N}}_u \cdot \boldsymbol{\sigma}_v = \hat{\mathbf{N}}_v \cdot \boldsymbol{\sigma}_u$.

8.3 The Weingarten map

The Gauss map takes a surface S to (part of) $\mathbb{S}^2$. How are their tangent spaces related? Recall that the mapping between two tangent spaces is just the ______________.

$$: T_{\mathbf{p}}S \to T_{\hat{\mathbf{N}}(\mathbf{p})}\mathbb{S}^2. \tag{8.4}$$

The tangent plane $T_{\hat{\mathbf{N}}(\mathbf{p})}\mathbb{S}^2$ contains all vectors perpendicular to the normal $\hat{\mathbf{N}}(\mathbf{p})$, which, by definition, is just __________. Thus the derivative of $\hat{\mathbf{N}}$ can be regarded as a map from $T_{\mathbf{p}}S$ onto itself:

$$\tag{8.5}$$

Let's see exactly how this derivative looks like.

Theorem 8.2. Let $\mathbf{p} \in S$. The derivative of the Gauss map satisfies

$$D_{\mathbf{p}}\hat{\mathbf{N}}(\boldsymbol{\sigma}_u) = \hat{\mathbf{N}}_u \qquad D_{\mathbf{p}}\hat{\mathbf{N}}(\boldsymbol{\sigma}_v) = \hat{\mathbf{N}}_v$$

(In other words, for the Gauss map, the differential-geometry concept of derivative reduces to the usual partial derivative.)

Proof: Let's prove the first equation: $D_{\mathbf{p}}\hat{\mathbf{N}}(\boldsymbol{\sigma}_u) = \hat{\mathbf{N}}_u$ (the other one is similar). Here are some facts from Chapter 6: Any $\mathbf{w} \in T_{\mathbf{p}}S$ can be thought of as a tangent vector to some curve $\boldsymbol{\gamma}(t)$, evaluated at say, at $t = 0$

$$\mathbf{w} = \tag{8.6}$$

By definition, the derivative $D_{\mathbf{p}}\hat{\mathbf{N}}$ gives the tangent of the mapped curve (Eq. 6.7).

$$D_{\mathbf{p}}\hat{\mathbf{N}}(\mathbf{w}) \equiv \tag{8.7}$$

Pick a point $\mathbf{p} = \boldsymbol{\sigma}(u_0, v_0)$. Consider the curve $\boldsymbol{\gamma}(u) = \boldsymbol{\sigma}(u, v_0)$ (a curve on which v=constant) undergoing the mapping $\hat{\mathbf{N}}$. The image of the tangent vector $\boldsymbol{\gamma}'(u) = \boldsymbol{\sigma}_u(u, v_0)$ is, by definition, the derivative of $\boldsymbol{\sigma}_u$. In symbols:

$$D_{\mathbf{p}}\hat{\mathbf{N}}(\boldsymbol{\sigma}_u) = \tag{8.8}$$

The RHS is an overly complicated way to express what is simply _______ (see Eq. (8.3)). $\square$

There's a special name for the (negative of) derivative of the Gauss map.

Definition. The *Weingarten*[2] *map* $W : T_{\mathbf{p}}S \to T_{\mathbf{p}}S$ is defined as

$$\tag{8.9}$$

For example, the previous theorem can be expressed in terms of the Weingarten map as:

$$\tag{8.10}$$

The minus sign is just a historical convention, just like the minus sign in $\kappa_s = -\dot{\mathbf{n}}_s \cdot \mathbf{t}$.

[2] *Julius Weingarten*(1836-1910), German mathematician who made important contributions to differential geometry. Note that in some books, the Weingarten map is called the *shape operator*.

8.4 The second fundamental form

Our train of thought so far:

- To quantify 'curvature' at a point on a surface, we measure the rate of change of the unit normal vector $\hat{\mathbf{N}}$, projected in some tangential direction (taking the queue from the formula $\kappa_s = -\dot{\mathbf{n}}_s \cdot \mathbf{t}$).
- The analogues of $\dot{\mathbf{n}}_s$ are $\hat{\mathbf{N}}_u$ and $\hat{\mathbf{N}}_v$.
- The analogue of $\mathbf{t}$ is any vector $\mathbf{w} \in T_{\mathbf{p}}S$, which is a linear combination of $\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v$.
- Thus, curvature could be measured by the four quantities:
- Alternatively, from Example 3, these quantities look like:
- In terms of the *Weingarten map*, W, these quantities look like:

In general, for a given $\mathbf{w} \in T_{\mathbf{p}}S$, the expression ______________ will contain contributions from all four curvature quantities at that point. We now show that $W(\mathbf{w}) \cdot \mathbf{w}$ can be expressed in a similar way to the first fundamental form.

Lemma 8.3. Let $\mathbf{p}$ be a point on surface S parametrized by $\boldsymbol{\sigma}(u, v)$. For any $\mathbf{w} \in T_{\mathbf{p}}S$, let $\mathbf{w}$ be the tangent to a curve $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ passing through $\mathbf{p}$. We have

$$W(\mathbf{w}) \cdot \mathbf{w} = L\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + 2M\frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}t} + N\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 \tag{8.11}$$

where $L =$ ______________, $M =$ ______________, $N =$ ______________.

Proof:

Equation 8.11 can be written in a similar way to the first fundamental form by defining

$$II \equiv W(\mathbf{w}) \cdot \mathbf{w}\ \mathrm{d}t^2 \tag{8.12}$$

Definition. The *second fundamental form* of the surface parametrized by $\boldsymbol{\sigma}(u, v)$ is defined as the expression.

$$II = \tag{8.13}$$

Information on the curvature of a surface is (partly) contained in the second fundamental form. Exactly how to fully extract this information remains to be seen...

Example 4. Calculate the second fundamental form of the following surfaces:

(a) $\boldsymbol{\sigma}(u, v) = \mathbf{a} + u\hat{\mathbf{p}} + v\hat{\mathbf{q}}$ (where $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ are constant orthonormal vectors)

(b) $\boldsymbol{\sigma}(u, v) = (\cos v, \sin v, u)$.

8.5 The Weingarten map in matrix form

Suppose Anton lives on the unit cylinder, and we have asked him to measure how much the cylinder curves at a point $\mathbf{p}$ on the cylinder. You can see that the answer depends on the route that Anton takes across $\mathbf{p}$.

Thus, we expect that there will be several definitions of surface curvature. We will now show that all types of curvatures can be fully extracted from the *Weingarten map*, which can be expressed as an elegant matrix. This is where the first and second fundamental forms come together in a grand conclusion!

Theorem 8.4. Let a surface S be parametrized by $\boldsymbol{\sigma}(u, v)$. The Weingarten map W can be expressed in matrix form as

$$W =$$

where the matrix represents a linear transformation wrt the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$. g and $\mathcal{F}_{II}$ are matrices of the coefficients of the first and second fundamental forms:

$$g = \qquad\qquad\qquad\qquad \mathcal{F}_{II} =$$

Proof. From Eq. 6.9, we know that $W : T_{\mathbf{p}}S \to T_{\mathbf{p}}S$ is a linear transformation (linearity inherited from the derivative), and can be represented by a matrix. To express W in the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$, let's see how it acts on each of the basis vectors. Suppose that

$$\begin{aligned} W\boldsymbol{\sigma}_u &= \qquad (*) \\ W\boldsymbol{\sigma}_v &= \end{aligned}$$

for some numbers a, b, c, d. With respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$, the above equations can be written as:

$$W \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

On the other hand, dotting equations $(*)$ with $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ yields four equations.

$\square$

Example 5. Find the Weingarten map W (in matrix form) for the plane and the cylinder in Example 4.

Example 6. Find a general expression for the matrix W in terms of E, F, G, L, M, N.

8.6 Types of curvatures

We have shown that information about the curvature can be represented by a matrix W which, as a linear transformation wrt the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$, looks fairly simple. What if we choose a different basis? Whatever quantity which represents the physical concept of "curvature" shouldn't depend on the choice of basis.

What properties of a matrix are unchanged by a change of basis? Linear algebra tells us that these are properties preserved by *similar matrices*, for example:

These are all good candidates for capturing the essence of surface curvature.

Definition. The *Gaussian curvature*, K, is defined as

(8.14)

Definition. The *mean curvature*, H, is defined as

(8.15)

Definition. The *principal curvatures*, κ_1 and κ_2, are defined as the ____________________ of W

These are not all independent quantities. Since K and H are invariant wrt to change of basis, we could diagonalise W and obtain the following result.

Corollary. The mean and Gaussian curvatures are related to the principal curvatures as follows:

Example 7. Find the Gaussian, mean and principal curvatures for the plane and the unit cylinder.

The previous Example gave us a glimpse into the meaning of these curvatures.

For instance, the cylinder has one circular direction ($\kappa = 1$) and one flat direction ($\kappa = 0$), so it makes sense that the *mean* curvature is indeed $1/2$.

The cylinder can be thought of as a rolled up plane, and $K = 0$ on both objects. This is because the Gaussian curvature is, in fact, an *intrinsic* property of a surface that is invariant under local isometry (*i.e.* length-preserving transformations like bending, rolling, twisting). We won't prove it here, but will simply state the result in the next section.

Example 8. Express H and K in terms of the fundamental-form coefficients E, F, G, L, M, N.

8.7 Concluding remarks

By now you should have a good idea of what *differential geometry* is all about. We have seen how geometrical properties of curves and surfaces can be studied using a combination of calculus, linear algebra and analysis.

Let's wrap up this course with a few remarks to tie up loose ends, and to give you a glimpse of what lies beyond this course. The results in this section are non-examinable.

8.7.1 Area interpretation of the Gaussian Curvature

Theorem 8.5. Suppose an area element $\mathrm{d}A$ on a surface S is mapped onto an area $\mathrm{d}\tilde{A}$ on the unit sphere via the Gauss map, $\hat{\mathbf{N}}$, then

$$|K|\ \mathrm{d}A = \mathrm{d}\tilde{A}$$

Proof. The area element on S parametrized by $\boldsymbol{\sigma}(u,v)$ is given by

$$\mathrm{d}A =$$

The Gauss map sends this area to $\mathrm{d}\tilde{A}$ on the unit sphere, where, from Theorem 8.2

$$\mathrm{d}\tilde{A} =$$

From the proof of Theorem 8.4, we find the following expressions for $\hat{\mathbf{N}}_u$ and $\hat{\mathbf{N}}_v$:

Amazingly, the Gaussian curvature is just a ratio of area elements post- and pre-Gauss map. From this result, you can see why $K = 0$ for the cylinder and plane: the area element post-Gauss map is zero (see the drawings we made in Example 1).

8.7.2 Gauss-Bonnet Theorem

Recall *Hopf's Umlaufsatz* for curves:

The signed curvature is a 'local' quantity (*i.e.* defined at a point on a surface), but the *total* signed curvature is clearly a *global* property of closed curves.

A similar link exists for surfaces. The Gaussian curvature K is a local property but the total Gaussian curvature.

is a global property of 'closed' surfaces. If we assume that the image of the Gauss map for a closed and bounded surface covers the entire unit sphere, so that we have

where we have used Theorem 8.5. This result is a special case of one of the most profound theorems in differential geometry: the *Gauss-Bonnet*[3] *Theorem*, which says:

Theorem 8.6. [*Gauss-Bonnet Theorem*] Let S be a compact surface, then

$$=$$

where $\chi = 2 - 2g$, and g is the __________ (the number of holes) on the surface.

The surface constant χ is called the ______________________________.

The Gauss-Bonnet theorem holds not only for 'smooth' surfaces, but also for solids such as a cube or a dodecahedron. This single theorem is the bridge that connects differential geometry (the LHS of the above quation) to *topology* ('rubber-sheet geometry') on the RHS.

[3] *Pierre Ossian Bonnet* (1819-1892), French mathematician famed for his contribution to differential geometry. He was a contemporary of Frenet and Serret.

8.7.3 Theorema Egregium

Every mathematician knows that the German mathematician *Carl Friedrich Gauss* (1777-1855) ranks amongst the top mathematical minds *ever*. Amongst his many illustrious and wide-ranging achievements, his 'remarkable theorem' (as he himself called it) is indeed one of the most remarkable. We state it here without proof.

Theorem 8.7. [*Gauss's Theorema Egregium*] The Gaussian curvature, K, can be expressed solely in terms of the first-fundamental-form coefficients E, F, G and their derivatives.

This means that *local isometries do not change Gaussian curvature*, *i.e.* one can regard K is a 'bending invariant' which is an *intrinsic* property of surface, like its area, its thickness or its density. Anton, for instance, will find that his measurement of K is the same before and after the incident at the end of Chapter 6.

In fact, K can be written explicitly in terms of E, F, G as

$$K = \frac{1}{(EG - F^2)^2}\left(\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix} \right). \tag{8.16}$$

This is sometimes called the 'Brioschi formula'. You can quickly verify that it works for the plane and cylinder.

8.7.4 A perfect map doesn't exist

Here is the final word on cartography, courtesy of the *Theorema Egregium*.

Lemma 8.8. There exists no geographical map of any portion of the Earth which is both equiareal and conformal.

Proof. Suppose there exists such a map. By Q1 of problem sheet 7, this map is a ____________________. This means the first fundamental forms are the same on the sphere and the plane. The *Theorema Egregium* implies that K is the same on these surfaces. However, $K =$ _____ on the plane, and $K > 0$ on the sphere (see problem sheet 8). A contradiction. □

8.7.5 The Fundamental Theorem of Surfaces

Theorem 8.9. [Fundamental Theorem of Surface Theory] Let E, F, G, L, M, N be any smooth real-valued functions of u and v.

[Existence] There exists a regular, orientable surface S, parametrized by $\boldsymbol{\sigma}(u, v)$ with the first and second fundamental forms given by

$$E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 \quad \text{and} \quad L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2.$$

[Uniqueness] The surface is unique up to translation and rotation.

8.7.6 Looking ahead

Many results we study in this course can be generalised to higher dimensions. A *manifold* is a topological space which looks like $\mathbb{R}^n$ locally (a surface is a 2-dimensional manifold). Differential geometry on higher-dimensional manifolds is the most fundamental tool in the studies of general relativity, particle physics, string theory and cosmology.

Interesting topics for further studies in an advanced course on differential geometry include *geodesics* (shortest routes on surfaces), *minimal surfaces* (what shape does a soap bubble form?) and *non-Euclidean geometry* (could parallel lines diverge?).

"Es ist nicht das Wissen, sondern das Lernen, nicht das Besitzen, sondern das Erwerben, nicht das Dasein, sondern das Hinkommen, was den größten Genuß gewährt."

It is not knowledge, but the learning; not the possession, but the acquiring; not the being-there, but the getting-there, that gives the greatest joy.

– Carl Friedrich Gauss