Analysis II

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These notes accompany the module *Analysis II* for first-year undergraduates reading mathematics at Warwick.

Results from *Analysis I* and *Foundations* will be assumed.

Some useful textbooks for this course:

Introduction to Real Analysis, Bartle R. G. and Sherbert D. R., 4th ed., Wiley (2011).

Guide to Analysis, Hart M., 2nd ed., Palgrave Macmillan (2001).

How to think about Analysis, Alcock L., Oxford University Press (2014).

I am also grateful for previous notes by James Robinson, Li Xuemei and David Mond.

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CHAPTER 1

CONTINUITY

We wish to give a mathematical definition of what it means for a function to be *continuous*. This is one of the most important concepts in analysis. In this course, we will focus on real-valued functions defined on $\mathbb{R}$. Extensions to functions defined on $\mathbb{C}$ and $\mathbb{R}^n$ will follow naturally in *Analysis III* and *Multivariable Calculus* next year.

At this point, continuity is still only an intuitive concept for most students. In particular, you may have been taught in school that the graph of a continuous function can be drawn without lifting the pen.

Example 1. Sketch the graph of a function (in the x-y plane) which fits with your intuition of a) a continuous function, b) a discontinuous function.

Although this seems to make a lot of intuitive sense, mathematics at university is all about ______________, so our first task is to define exactly what it means for a function to be continuous, using mathematical symbols without any drawings.

1.1 The ε-δ definition of pointwise continuity

Consider a real-valued function f defined on a subset A of $\mathbb{R}$. In symbols, we write ____________________. Recall that A is called the _______________ and B the _______________.

Looking at your discontinuous example, you might agree that it may be better to start defining continuity ____________________, say, __________ $\in A$.

The mathematical definition of pointwise continuity which has become standard today was first published by Bolzano[1] in 1817.

Definition. A function $f : A \to \mathbb{R}$ is said to be *continuous* at $x_0 \in A$ if

This is one the most important definitions you will study in your time at university. However, at the moment this definition probably won't make much sense to you, so let's try to unpack what it says.

The intuitive idea behind continuity at the point $x = x_0$ is that we can't have any gaps in the y values __________ $f(x_0)$ (let's call this y value y_0). Let's make a quick sketch.

[1]Bernhard Bolzano (1781-1848), a Bohemian mathematician, logician and Catholic priest, best remembered today for his important contributions to analysis.

If f is continuous at x_0, it makes sense to demand that we can always find y values ______________ close to $y_0 = f(x_0)$. Symbolically, the set of "*all y values arbitrarily close to y_0*" can be expressed as follows.

Definition. Let $\varepsilon > 0$ and $y_0 \in \mathbb{R}$. The *ε-neighbourhood* of y_0 is the set:

Occasionally, we denote the above neighbourhood as ____________.

Now, we want the y values inside the ε-neighbourhood of y_0 to "come from" some values of x. In other words, there should be some x values such at that $y =$ _______. It's natural to demand that those values of x should also be in some neighbourhood of x_0. We are satisfied as long as "________________ *such a neighbourhood*" in the domain A. This statement can be written symbolically as:

Combining what we have discussed so far, we say that a function f is continuous at $x = x_0$ if there exists a neighbourhood of x_0 which gets mapped by f into an arbitrarily small neighbourhood of $y_0 = f(x_0)$. Symbolically, we write:

This explains the intuition behind the definition. I strongly encourage you read this many times over, and continue to think about it obsessively! Some remarks:

- Do say: f is continuous. Don't say: $f(x)$ is continuous.
- The ordering of the quantifiers $\forall\varepsilon$ and $\exists\delta$ cannot be exchanged in the ε-δ definition. In general, δ will depend on ε (think about why this makes intuitive sense).
- If you can find a certain value of δ that works, then so will any other values δ' such that $0 < \delta' < \delta$. Can you see why this is true? Often you will find that there is usually a largest value of δ that works for each choice of ε. See Quiz 1.

1.2 Examples

We can use the ε-δ definition to prove that a given function is continuous at a given point (although we will see later how this can often be avoided). This kind of proof (also called proof from *first principles*) may seem a bit tricky at first, but with practice you will soon get the idea behind the ε-δ game, which is essentially a hunt for an expression for δ in terms of ε that would fit *precisely* with the definition of continuity. There are infinitely many possible answers!

Example 2. Prove that the following functions $f\colon \mathbb{R} \to \mathbb{R}$ are continuous at $x = 0$.
a) $f(x) = 3x$ b) $f(x) = ax + b$ (for some $a, b \in \mathbb{R}$.) c) $f(x) = x^2 + 1$.

[Can you see other possible values of δ that work for these proofs?]

Sometimes more tricks are needed. For beginners, I recommend working on a draft proof before writing it out properly.

Useful inequalities: Let $a, b, c \in \mathbb{R}$, we have the following:

$|a| < 1 \implies$ $|a| > 1 \implies$

$a < b$ and $b < c \implies a <$ $a < b$ and $a < c \implies a <$

Triangle inequality:

Reverse triangle inequality:

Example 3. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous at $x = 1$.

[*Hint: If we're stuck, we should recall that the smaller the* δ *the better chance we have of making the proof work, so it may help to try to put an upper bound on* δ*.*]

Beware of denominators! Always check if they could be zero and mitigate accordingly.

Example 4. Prove that $f\colon$ ________ $\to \mathbb{R}$ defined by $f(x) = 1/x$ is continuous at all $x \neq 0$.

1.3 Algebra of continuous functions

ε-δ proofs can get very tedious. It would save a lot of work if we can show that the sum, product and composition of continuous functions are again continuous. But first:

Question. For $f\colon A \to \mathbb{R}$, suppose that $\forall \varepsilon > 0, \exists \delta > 0$ such that for all $x \in A$, $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon/2$. Is f continuous at x_0?

Theorem 1.1. If $f, g\colon A \to \mathbb{R}$ are both continuous at $x_0 \in A$, then the function h defined by

$$h(x) = f(x) + g(x)$$

is also continuous at x_0.

[Study the proof in Quiz 1 and reproduce it here.]
Proof:

Theorem 1.2. If $f, g\colon A \to \mathbb{R}$ are both continuous at $x_0 \in A$, then the function h defined by

$$h(x) = f(x)g(x)$$

is also continuous at x_0.

Proof:

When it comes to function compositions, I always find that a sketch (of what function maps which domain to where) helps.

Theorem 1.3. Let $f\colon A \to \mathbb{R}$ be continuous at $x_0 \in A$. Let $g\colon B \to \mathbb{R}$ be continuous at $f(x_0)$, and $f(A) \subseteq B$, then the function h defined by

$$h(x) = g \circ f\,(x) = g(f(x))$$

is also continuous at x_0.

[Study the proof in Quiz 1 and reproduce it here.]
Proof:

Theorem 1.4. If $f\colon A \to \mathbb{R}$ is continuous at $x_0 \in A$ and $f(x_0) \neq 0$, then

(a) $1/f$ is a well-defined function in some neighbourhood $N_\delta(x_0) \subset A$,

(b) $1/f$ is continuous at x_0.

Proof:

In summary, the Algebra of Continuous Functions refers to the fact that pointwise continuity is preserved under addition, multiplication, division and composition of functions (T&Cs apply). Here's a nice consequence of the AoCF.

Corollary. Functions of the form $P(x)/Q(x)$ (where $P(x)$ and $Q(x)$ are polynomials) are continuous at all $x \in \mathbb{R}$ where ____________.

You should try writing down a chain of justification and identify what results are needed to prove the above result.

1.4 Discontinuity

How can we prove that a function is discontinuous at a point? We will study two methods in this section.

1.4.1 Method I: Negating the definition

First let's do a bit of logic revision. Let $P(\delta)$, $Q(\varepsilon)$ and $R(\varepsilon, \delta)$ be statements which depend on variables ε, δ.

Example 5. Negate (say the opposite) of the following statements.

(a) $\forall \varepsilon, Q(\varepsilon)$.

(b) $\exists \delta : P(\delta)$.

(c) $P \implies Q$.

(d) $\forall \varepsilon, \exists \delta : R(\varepsilon, \delta)$.

Now consider a function f. The continuity of $f\colon A \to \mathbb{R}$ at x_0 can be written as the following logical statement:

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \; P(\delta, x) \implies Q(\varepsilon, x)$$

where $P(\delta, x)$:

$Q(\varepsilon, x)$:

Negating the above statement, we find the following.

Lemma 1.5. A function $f\colon A \to \mathbb{R}$ is said to be *discontinuous* at $x_0 \in A$ if

Let's try to interpret this result graphically. The Lemma simply says that a function is discontinuous at x_0 if, in an arbitrary neighbourhood of x_0, we can find an x whose image 'jumps' outside the ε neighbourhood of $f(x_0)$.

Note that the ordering of the logical quantifiers means that the jump distance ε is fixed for all values of δ. In other words, no matter how closely you zoom into x_0, the function value always jumps by a distance of at least ε.

Example 6. Prove that the step function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0. \end{cases}$$

is discontinuous at $x = 0$.

For this example you will need to recall some results from Analysis I. In the *Completeness* chapter, you proved that between any two distinct real numbers, one can find (infinitely many) rational and irrational numbers.

Example 7. Prove that the Dirichlet function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

is discontinuous at all $x_0 \in \mathbb{R}$.

1.4.2 Method II: The sequential criterion

This technique to prove discontinuity is based on the following theorem.

Theorem 1.6. (Sequential criterion for continuity). A function $f\colon A \to \mathbb{R}$ is continuous at $x = c \in A$ if and only if, for every sequence (x_n) in A converging to c, we have ___________________.

To prove that a function is discontinuous at $x = c$, we just need to produce a sequence (x_n) such that $x_n \to c$ but $f(x_n) \not\to f(c)$. The theorem implies that f cannot be continuous.

Can you see why we can't easily use the sequential criterion to prove that a function is *continuous* (rather than discontinuous)?

We will prove the "only if" part of the Theorem (the "if" part is in this week's Quiz).

Example 8. Redo Example (6) using the sequential criterion

Example 9. Redo Example (7) using the sequential criterion

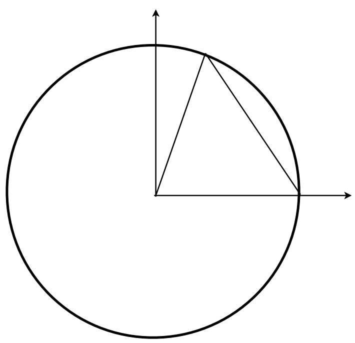
1.5 Trigonometric functions

Finally we consider why trigonometric functions are continuous.

Lemma 1.7. If $x \in (0, \pi/2)$, we have

$$\sin x < x.$$

Proof: Consider a unit circle in which a sector subtends an angle x radian as shown in the figure below.



For the case $x \in (-\pi/2, 0)$, because both x and $\sin x$ are odd, the above Lemma implies $|\sin x| < |x|$. Using the periodicity of sine, we can extend the Lemma to $x \in \mathbb{R}$.

Corollary. $\forall x \in \mathbb{R}$, $|\sin x| \le |x|$.
Perhaps we can see this more clearly by sketching some graphs (but I encourage you to try writing a rigorous proof).

Now we can prove the the sine function is continuous at every point on $\mathbb{R}$. The proof we will study relies on a well-known sum-to-product formula (which you should be able to derive): $\forall \alpha, \beta \in \mathbb{R}$,

$$\sin \alpha - \sin \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right).$$

Theorem 1.8. The function $f(x) = \sin x$ is continuous at every $x \in \mathbb{R}$.

Example 10. Deduce that the function $f(x) = \cos^2(3 - 4x)$ is continuous at all $x \in \mathbb{R}$.

Some final remarks.

- Since $\cos x = \sin(\pi/2 - x)$ and $\tan x = \sin x / \cos x$, the cosine and tangent functions are continuous (where defined), thanks to the algebra of continuous functions. We will deal with the continuity of the exponential and log later on when we study power series.
- This chapter establishes rigorously the meaning of continuity at a *point*. Next, we will study special properties of functions that are continuous on an *interval*.

Interlude: open and closed sets

Before we begin the next chapter, we need to understand precisely what closed and open sets are. You will meet these concepts again when you study *metric spaces* next year.

Definition. A set $A \subseteq \mathbb{R}$ is said to be open if

In other words, a set A is open if at every point in A, we can find a neighbourhood which is contained in A.

Definition. A set $A \subseteq \mathbb{R}$ is closed if ________________________.

Likewise, if A is open, it follows that $\mathbb{R} \backslash A$ is closed. In other words, the ______________ of an open set is closed, and vice versa. You should try to prove this using sets and logic from Foundations.

Warning: A set can be open, closed, both or neither. A set which is not open is not necessarily closed!

Example 11. Prove that the interval (a, b) is open.

Example 12. Prove that $(a, b]$ is not open.

Example 13. Prove that $\mathbb{R}$ and $\emptyset$ are both open and closed.

Example 14. If A and B are both open, show that $A \cap B$ and $A \cup B$ are open.

You can show (say, by induction) that any finite number of intersections/unions of open sets is still open. In fact, the union of countably *infinite* number of open sets is still open. However, the intersection of infinitely many sets is not necessarily open. Try to think of an example.

Corollary. If A and B are closed, show that $A \cap B$ and $A \cup B$ are closed.

Example 15. Prove that $[a, b]$ is closed.

CHAPTER 2

CONTINUITY ON AN INTERVAL

We now discuss what it means for a function to be continuous over an interval rather than a point. In particular, we will study special properties of functions that are continuous on *closed, bounded* intervals.

Example 1. Give an example of a function which is (intuitively) continuous on an interval I, but not outside I, where (a) $I = [-1, 1]$, (b) $I = (-1, 1)$.
Does the ε-δ definition make sense at $x = \pm 1$ for Example (a)?

We see that the ε-δ definition can still be used even at the endpoints of the closed, bounded interval $[a, b]$.

Here is a useful result that we will need in this chapter. Study the proof in the Quiz.

Lemma 2.1. (*Sign-preservation lemma*). Suppose a function $f\colon A \to \mathbb{R}$ is continuous at $x = x_0$.

- If $f(x_0) > 0$ then $\exists \delta > 0$ such that $\forall x \in A$,
- If $f(x_0) < 0$ then $\exists \delta > 0$ such that $\forall x \in A$,

Proof:

In other words, if f is continuous at a point x_0, we can find an interval containing x_0 in which the sign of $f(x)$ is fixed.

Question. Suppose f is continuous on $[0, 2]$. What can we deduce from the sign-preservation lemma if each of the following holds? (a) $f(1) < 0$, (b) $f(2) > 0$.

2.1 The Intermediate-Value Theorem (IVT)

Theorem 2.2. (*Almost the IVT but not quite*) Let f be a function which is continuous on $[a, b]$. If $f(a) < 0$ and $f(b) > 0$, then there exists $c \in (a, b)$ such that ______________.

It's helpful to use sketches to help understand what the theorem says, and to make sense of the steps in the following proof.

Proof: Consider the set

$$S = \{x \in [a, b] : f(x) \leq 0\}.$$

We claim that S is a non-empty subset of $\mathbb{R}$ which is bounded above. This is because:

- _____ $\in S$ (non-empty)
- $\forall x \in S, x \leq$ _____ (bounded above)

The ____________________ property of $\mathbb{R}$ implies that S has a ____________________, which we will call c. Note also that $c \in [a, b]$ (why?)

The ____________________ Law states that only one of the following is true: $f(c) > 0$, $f(c) < 0$, or $f(c) = 0$. We argue that the first two cases lead to a contradiction.

Case I: Suppose $f(c) > 0$. Clearly $c \neq a$ since $f(a) < 0$, which means $c \in (a, b]$ (it's still possible that $c = b$).

Since f is continuous at c, the sign-preservation lemma implies that $\exists \delta > 0$ such that

____

On the other hand, since c is the least upper bound of S, $c - \delta$ is not an upper-bound, and thus $\exists x \in S$ such that

$$c - \delta < x \leq c.$$

In other words, $\exists x$____. A contradiction.

Case II:

In particular, $f(c + \delta/2) < 0$, meaning that $c + \delta/2$ is a member of S. Since c is an upper bound of S,

$$\leq$$

This contradicts the fact that _______________.

Therefore, we must have $f(c) = 0$. This clearly rules out both $c = a$ and $c = b$, and so $c \in (a, b)$. $\square$

Theorem 2.3. (*Intermediate-Value Theorem*) Let g be a function which is continuous on $[a, b]$. If v is a value strictly between $g(a)$ and $g(b)$, then there exists $c \in (a, b)$ such that ______________.

Proof. If $g(a) < v < g(b)$, apply Theorem 2.2 to the function $f(x) \equiv g(x) - v$.
If $g(b) < v < g(a)$, apply Theorem 2.2 to the function $f(x) \equiv$____________. □

Here's an easy way to remember the IVT: A continuous function g on $[a, b]$ takes __________________________________ between $g(a)$ and $g(b)$.

One of the most useful applications of the IVT is to prove that an equation (containing continuous functions) has a solution in a certain closed interval.

Example 2. Does the polynomial $g(x) = x^6 + 11x + 7$ have any real roots?

Example 3. Study what happens when we try to apply the IVT to $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Example 4. Does the equation $2 \sin x = x^2 - 1$ have any solutions?

Some remarks on the IVT.

- One the most important applications of the IVT is *root finding*. One such method (the *bisection method*) is discussed in the Quiz.
- A fun application of the IVT is that you can sometimes 'cure' a wobbly table simply by rotating it. See: `https://www.youtube.com/watch?v=OuF-WB7mD6k`.

2.2 The Boundedness Theorem

The following definition of boundedness is very similar to that of a bounded sequence from Analysis I.

Definition. A function $f\colon A \to \mathbb{R}$ is said to be *bounded on A* if

Definition. A function $f\colon A \to \mathbb{R}$ is said to be *unbounded on A* if

Example 5. Determine whether the following functions are bounded or unbounded on the interval $(0, 1)$. a) $f(x) = \sin(x + 5)$, b) $f(x) = 1/x$.

Can you think of a continuous function on a *closed bounded* interval which is unbounded? We will now prove that there is no such function. The proof requires these results from Analysis I.

Lemma 2.4. Every convergent sequence of real numbers is bounded.

Lemma 2.5. If (x_n) is a convergent sequence and $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then

Theorem 2.6. (*Boundedness Theorem*) Let I be a closed bounded interval $[a, b]$. If $f\colon I \to \mathbb{R}$ is a function which is continuous on I, then ________________________.

Proof. Suppose f is not bounded on I (and let's work towards a contradiction).

Since f is not bounded on I, for every $n \in \mathbb{N}$, $\exists x_n \in I$ such that ____________. Consider the sequence (x_n). Observe that (x_n) is a bounded sequence (since each $x_n \in [a, b]$, a bounded interval). By the _____________________________ Theorem, $\exists$ a subsequence (x_{n_i}) of (x_n) that converges to a number x. By lemma 2.5, _______________.

Since f is continuous at x, $f(x_{n_i})$ converges to $f(x)$ by the _________________________. Lemma 2.4 implies that the sequence $(f(x_{n_i}))$ is ____________.

But this contradicts the fact that the sequence $(f(x_{n_i}))$ is *unbounded*, because

□

It is important to note that all the assumptions are essential to make the Boundedness Theorem work, namely:

1) I must be closed. 2) I must be bounded. 3) f must be continuous on I.

Question. Think of an unbounded function on I (if any) when each of the above conditions is not satisfied.

Example 6. Let $f\colon [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$. Are the following functions bounded on $[a, b]$?

a) $g(x) = f(x) + 1$, b) $h(x) = f(x + 1)$

The Boundedness Theorem can be made even more precise by the following Theorem.

2.3 The Extreme-Value Theorem (EVT)

Definition. The function $f\colon A \to \mathbb{R}$ has an ______________________ on A if there exists a point $x^* \in A$ such that, for all $x \in A$, $f(x^*) \geq f(x)$.

We call x^* an absolute maximum _________ of f in A.

Definition. The function $f\colon A \to \mathbb{R}$ has an *absolute minimum* on A if

Definition. An absolute maximum or minimum is also called an *absolute extremum.*

Example 7. Determine whether the function defined by $f(x) = 1/x$ has an absolute extremum on the following interval. a) $[1, 2]$, b) $(0, 1)$.

Theorem 2.7. (*Extreme-Value Theorem*) Let $f\colon I \to \mathbb{R}$ be a function which is continuous on a closed bounded interval I. Then f has an absolute maximum and an absolute minimum on I.

The proof has the same flavours as that of the IVT and the Boundedness Theorem.

Proof. Consider the image of I under f:

$$f(I) =$$

The Boundedness Theorem states that $f(I)$ is a bounded subset of $\mathbb{R}$. Clearly $f(I)$ is also non-empty, so the __________________ of $\mathbb{R}$ implies that $f(I)$ has a supremum and an infimum. Let's call them:

$$s^* = \sup f(I) \qquad \text{and}$$

First, we wish to find $x^* \in I$ such that $s^* =$ _________.

Since s^* is the least upper bound of $f(I)$, the number $s^* - 1/n$ (for all $n \in \mathbb{N}$) is not an upper bound of $f(I)$, meaning that

$(*)$

The sequence (x_n) is bounded (because I is bounded), and by the Bolzano-Weierstrass theorem, $\exists$ a subsequence (x_{n_i}) converging to some number x^*. By Lemma 2.5, __________.

Since f is continuous on I, the sequential criterion implies that ____________________.
From $(*)$, we also deduce that for all $i \in \mathbb{N}$,

$$s^* - \frac{1}{n_i} < f(x_{n_i}) \leq s^*$$

Now take the limit as $i \to \infty$, the _________________________ implies that

In other words, x^* is the absolute maximum point of f on I. The existence of the absolute minimum point x_* can be similarly proved. (Alternatively, we could also apply what we proved to find the absolute maximum point of the function $g = -f$.) $\square$

The EVT is generally useful as a bounding tool when we need a quick upper or lower bound for continuous functions on a closed bounded interval.

Example 8. Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ with $f(x) > 0$ for all $x \in [a,b]$. Prove that there exists a constant $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in [a,b]$.

CHAPTER 3

LIMITS

3.1 Limits of functions

In Analysis I, we studied what it means precisely when we take a limit of a sequence. The meaning of the expression $\lim_{n\to\infty} x_n = x$ was made precise using N and ε. In this Chapter, we will study limits of *functions* and see what $\lim_{x\to c} f(x) = L$ means precisely. Many key concepts in analysis can be expressed in terms of limits, including continuity (Thm 3.1), differentiation (Ch. 5) and integration (*Analysis III*).

Definition. Let $c \in (a, b)$ and $f\colon (a, b) \setminus \{c\} \to \mathbb{R}$. We write:

$$\lim_{x\to c} f(x) = L,$$

if

This should remind us of the definition of continuity at c, but note the following.

- f is not assumed to be continuous anywhere.
- f does not even have to be defined at $x = c$. This explains the strange domain of f (although later on we may choose to include c in the domain).
- The limit definition makes no mention of what happens *at* $x = c$, but only what happens *close to* c. This explains why the δ-neighbourhood of c is ______________.

If $\lim_{x \to c} f(x) = L$, we say that "*the limit of f at c is L*" or "f ______________ to L at c." Sometimes we also write "__________________ as $x \to c$".

The next Theorem shows us that pointwise continuity can be defined in terms of limits.

Theorem 3.1. $f\colon A \to \mathbb{R}$ is continuous at $c \in A$ iff $\lim_{x \to c} f(x) = f(c)$.

Proof.

The above theorem means that you can only start substituting values into $f(x)$ when you are certain that $f(x)$ is continuous at $x = c$.

Example 1. Evaluate: a) $\lim_{x \to 0} x \sin x$ b) $\lim_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$.

3.2 Sequential criterion for limits

Here is the limit version of Theorem 1.6 (sequential criterion for continuity).

Theorem 3.2. Let $c \in (a, b)$ and $I = (a, b) \setminus \{c\}$. Let $f : I \to \mathbb{R}$. The following are equivalent.

(a) $\lim_{x \to c} f(x) = L$

(b) For every sequence $x_n \in I$ converging to c, we have _______________.

Proof.

You should certainly compare the above results to Theorem 1.6, which we often used to prove that a function is *discontinuous*. Similarly, the limit version of the theorem gives us a nice way to show that a limit *does not exist*, *i.e.* by EITHER

(i) finding a sequence $(x_n) \to c$ but ____________________, OR,

(ii) finding two sequences (x_n) and (y_n) converging to c, but ____________________.

Method (ii) works because if a limit (for sequences) exists, then it is unique.

Example 2. Prove that the following limits do not exist. a) $\lim_{x\to 0} \frac{1}{x}$ b) $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$.

[*Note: If a limit of* f *at* c *does not exist, we can also say that* f ______________ *at* c.]

3.3 Limit theorems

Theorems 3.1 and 3.2 show us that results for limits can be cast in terms of either continuous functions or sequences. Happily, this means that we have (more or less) already proved all of the following results.

Theorem 3.3. [____________________________] Let $c \in (a, b)$ and $I = (a, b) \setminus \{c\}$. Let $f, g\colon I \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

(a)

(b) If $M \neq 0$, then

The proofs for the *algebra of limits* are almost identical to those for the algebra of continuous functions. In addition, we can also construct a similar proof for:

Theorem 3.4. [*Limit of composite functions*] Let $I = (a, b) \setminus \{c\}$ and $J = (u, v) \setminus \{d\}$. Consider functions $f\colon I \to J$ and $g\colon J \to \mathbb{R}$. If $\lim_{x\to c} f(x) = d$ and $\lim_{y\to d} g(y) = M$, then

$$\lim_{x\to c} g \circ f(x) = \underline{\qquad}.$$

We often use a special case of the above theorem when g is *continuous* at d. We can then 'pass' the limit into g, or think of it as a change of variable $y = f(x)$.

Lemma 3.5. [*Change of variable*] Let $I = (a, b) \setminus \{c\}$ and $J \subseteq \mathbb{R}$. Consider functions $f\colon I \to J$ and $g\colon J \to \mathbb{R}$. If $\lim_{x\to c} f(x) = d$ and g is continuous at d, then

$$\lim_{x\to c} g \circ f(x) =$$

Example 3. Evaluate $\lim_{x\to \pi/2} \sin\cos(x)$

Theorem 3.6. [$\underline{\qquad\qquad\qquad\qquad\qquad}$] If $f(x) \le g(x) \le h(x)$ and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = \alpha,$$

then $\underline{\qquad\qquad\qquad\qquad}$.

(This follows from the same result for sequences).

Theorem 3.7. [*Limits preserve non-strict inequalities*] Let $c \in (a, b)$ and $I = (a, b) \setminus \{c\}$. Let $f\colon I \to \mathbb{R}$. Then, for all $x \in I$,

$$\alpha \le f(x) \le \beta \implies$$

(This follows from a similar preservation theorem for sequences).

Here are two interesting applications of the above limit theorems.

Theorem 3.8. [*A very useful limit*] $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

Proof. The geometric proof of lemma (1.7) shows that, for $x \in (0, \pi/2)$, we have:

Occasionally you may come across the function

$$\operatorname{sinc} x =$$

This is a continuous function on $\mathbb{R}$ with many real-world applications (*e.g.* signal processing). You should definitely be familiar with its graph. You will meet the sinc function again when you study complex analysis and the Fourier transform.

Example 4. Evaluate $\lim_{x \to 0} \frac{1 - \cos x}{x}$.

[*Tip: When working with limits, It's a good habit to indicate where limit theorems have been used.*]

3.4 One-sided limits

By restricting to one side of the punctured neighbourhood in the definition of limits, we can define *one-sided* limits:

Definition. [*One-sided limits.*]

(a) We say that $f\colon (a,c) \to \mathbb{R}$ has a *left-hand limit* L at c, and write

$$\lim_{x \to c^-} f(x) = L,$$

if

(b) We say that $f\colon (c,b) \to \mathbb{R}$ has a *right-hand limit* M at c, and write

From the definitions, we can immediately deduce the following results:

Lemma 3.9. Let $c \in (a,b)$ and $f\colon (a,b) \setminus \{c\} \to \mathbb{R}$. Then $\lim\limits_{x \to c} f(x) = L$ iff $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x) = L$.

Theorem 3.10. [*Continuity in terms of one-sided limits*] Let $c \in (a,b)$ and $f\colon (a,b) \to \mathbb{R}$. Then f is continuous at c iff $\lim\limits_{x \to c^+} f(x) = \lim\limits_{x \to c^-} f(x) = f(c)$

Example 5. Write down the one-sided limits of the functions at $x = N \in \mathbb{Z}$.

a) $\lceil x \rceil$, b) $x - \lfloor x \rfloor$

(*You should try to prove these using the definitions.*)

3.5 Limits at infinity

We are used to saying "$1/x$ tends to 0 as x tends to infinity", but we can now make this sentence precise using the language of analysis.

Definition. [*Limits as* $x \to \pm\infty$.]

(a) We say that $f\colon (a,\infty) \to \mathbb{R}$ has a *limit* L *as* $x \to \infty$, and write

$$\lim_{x\to\infty} f(x) = L,$$

if

(b) We say that $f\colon (-\infty,a) \to \mathbb{R}$ has a *limit* M *as* $x \to -\infty$, and write

These should remind you of the ε, N definition for the limit of a sequence.

Example 6. Prove, using the definition, that $\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0$.

It is worth noting that the squeeze theorem still works as $x \to \pm\infty$. You can guess that the proof is very similar to the version for sequences.

Theorem 3.11. [*Squeeze theorem - version for limits at* ∞]
If $f(x) \leq g(x) \leq h(x)$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = \alpha$, then ___________________

3.6 Infinite limits (when functions blow up)

We wish to give precise meaning to statements like "$1/x^2$ *blows up at* $x = 0$."

Strong warning: The limit $\lim_{x\to 0} 1/x^2$ ________________________, but we can still use the symbol $\pm\infty$ to give a sense of the direction in which the function 'blows up'. Never treat the symbol ∞ like a number, and always feel nervous about writing it down at all. Most importantly, never write "$\ldots = \infty$" unless there's a limit on the LHS.

Definition. [*Infinite limits.*]

(a) We say that $f\colon (a,b) \setminus \{c\} \to \mathbb{R}$ *tends to* ∞ as $x \to c$ and write

$$\lim_{x\to c} f(x) = \infty,$$

if

(b) We say that $f\colon (a,b) \setminus \{c\} \to \mathbb{R}$ *tends to* $-\infty$ as $x \to c$ and write

These definitions should by now feel like a natural way to express the sense of unboundedness near $x = c$ in the language of analysis.

We can also have *one-sided* infinite limits. You should try writing down the definitions of these expressions.

$$\lim_{x\to c^-} f(x) = \infty \qquad \lim_{x\to c^-} f(x) = -\infty \qquad \lim_{x\to c^+} f(x) = \infty \qquad \lim_{x\to c^+} f(x) = -\infty$$

Finally, we can combine infinite limits with limits at ∞, giving precise meaning to statements like "$f(x) = x$ tends to ∞ as $x \to \infty$."

Definition. [*Infinite limits as $x \to \infty$.*] We say that $f\colon (a, \infty) \to \mathbb{R}$ *tends to ∞ as $x \to \infty$*, and write

$$\lim_{x\to\infty} f(x) = \infty,$$

if

Example 7. Prove that for any odd integer $n \geq 1$, $\lim_{x\to\infty} x^n = \infty$, and $\lim_{x\to-\infty} x^n = -\infty$.

3.7 Summary

You must be able to state the definitions of all of the following types of limits.

- Finite limits: $\lim_{x\to c} f(x) = L$
- One-sided limits: $\lim_{x\to c^-} f(x) = L$, $\lim_{x\to c^+} f(x) = L$
- Limits at infinity: $\lim_{x\to\infty} f(x) = L$, $\lim_{x\to-\infty} f(x) = L$
- Infinite limits: $\lim_{x\to c} f(x) = \infty$, $\lim_{x\to c} f(x) = -\infty$
- Infinite one-sided limits: $\lim_{x\to c^-} f(x) = \infty$, $\lim_{x\to c^-} f(x) = -\infty$, $\lim_{x\to c^+} f(x) = \infty$, $\lim_{x\to c^+} f(x) = -\infty$
- Infinite limits at infinity: $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = -\infty$

CHAPTER 4

INVERSE FUNCTIONS

You may remember the inverse function from school as the reflection of the curve $y = f(x)$ about the line $y = x$. But at university, we start from definitions and build precise vocabulary around the idea of inverse functions (without relying on pictures for definition). You may recall the following definitions from Foundations.

Definition. Consider the function $f: A \to B$.

- f is *injective* if
- f is *surjective* if
- f is *bijective* if

Definition. If $f: A \to B$ is a bijection, its inverse ______________ is defined by

$$f^{-1}(y) = x \qquad \text{if}$$

Definition. Consider the function $f: A \to B$.

- $f(A)$ is said to be the ____________ of A under f (just another way to say that $f(A)$ is the range of f).
- $f^{-1}(B)$ is said to be the ______________ of the set B under f (another way to say that $f^{-1}(B)$ is the domain of f).

It's useful to note that $f: A \to f(A)$ is clearly a ______________ function.

4.1 Monotone functions

Definition. Consider the function $f\colon A \to B$. For all $x_1, x_2 \in A$,

- f is *increasing* on A if
- f is *strictly increasing* on A if
- f is *decreasing* on A if
- f is *strictly decreasing* on A if
- f is *monotone* on A if
- f is *strictly monotone* on A if

The main goal of this Chapter is to prove the following theorem.

Theorem 4.1. [________________________________] If $f\colon I \to f(I)$ is a continuous, strictly monotone function on an interval I, then $f^{-1}\colon f(I) \to I$ exists, and is also continuous and strictly monotone (in the same sense as f) on the interval $f(I)$.

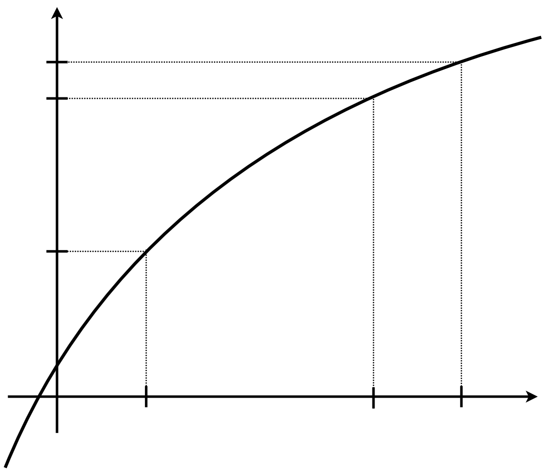
This useful theorem allows us to "invert" any continuous functions on an interval over which it is strictly monotone. This is how we define, for example, the function $\sin^{-1}$.

Here's a lemma which will be useful in the proof. Study its proof in the Quiz.

Lemma 4.2. [__________________________] Let $I \subseteq \mathbb{R}$ be an interval and let $f\colon I \to \mathbb{R}$ be continuous on I. Then $f(I)$ is also an interval.
Proof:

Lemma 4.3. If $f\colon [a,b] \to [f(a), f(b)]$ is a surjective and strictly increasing function on $[a,b]$, then f is continuous on (a,b).

Proof:



With slight changes, the proof can be modified to show the following: a) f is in fact continuous on $[a, b]$, b) f is in fact continuous on any interval $I \subseteq \mathbb{R}$.
Let's state this generalisation here.

Lemma 4.4. If $f\colon I \to f(I)$ is a strictly increasing function on I (where I and $f(I)$ are intervals), then f is continuous on I.

Lemma 4.5. If $f\colon I \to f(I)$ is a continuous, strictly increasing function on an interval I, then $f^{-1}\colon f(I) \to I$ exists, and is also strictly increasing on the interval $f(I)$.
Proof:

Theorem (*Inverse-function theorem*)**.** If $f\colon I \to f(I)$ is a continuous, strictly monotone function on an interval I, then $f^{-1}\colon f(I) \to I$ exists, and is also continuous and strictly monotone (in the same sense as f) on the interval $f(I)$.

Proof. Consider $f\colon I \to f(I)$, which is a continuous, strictly increasing function. (If f is strictly decreasing, consider $g(x) =$ __________ instead.)

- Since I is an interval, $f(I)$ is an interval by Lemma ________.
- $f^{-1}\colon f(I) \to I$ exists and is strictly increasing by Lemma ________.
- Therefore, Lemma ________ applies and thus f^{-1} is continuous on $f(I)$. □

CHAPTER 5
DIFFERENTIATION

In your previous encounter with calculus, you would have gained dexterity with a variety of differentiation techniques. However, in Analysis, we will strip everything back to definitions and build everything up again with a stronger foundation that we have gained through the previous chapters.

Definition. Let $f\colon (a, b) \to \mathbb{R}$ and let $c \in (a, b)$. The ______________ of f at c, denoted $f'(c)$, is defined as:

$$=$$

f is said to be ____________________ at c if the limit above exists (*i.e.* is finite).

The derivative can also be written as:

$$=$$

In this course, we will primarily regard the derivative as a __________, as opposed to a rate of change, or the gradient of a tangent to a curve.

Convention: We will not be using the symbol dy/dx.

Example 1. Find the derivative of $f(x) = x^2$ at $x = c$.

It is often convenient to regard the derivative as a function of x in its own right, where the domain of f' is a subset of the domain of f (*i.e.* wherever f is differentiable). Thus, in the previous example, we can also say that the derivative of f is a function $f' \colon \mathbb{R} \to \mathbb{R}$ where $f'(x) = 2x$.

Example 2. Find the derivative of the function $f(x) = \sin x$ defined on $\mathbb{R}$, at $x = c$.

Theorem 5.1. If $f \colon (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then f is continuous at c.
Proof:

Example 3. Investigate the differentiability of the function $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$.

Using one-sided limits, we can easily determine whether a function is differentiable or not at $x = c$. Using Lemma 3.9, we deduce that a function f is differentiable at c iff

$$\text{where} \quad F(x) \equiv \frac{f(x) - f(c)}{x - c}.$$

Example 4. Demonstrate, using the function $f(x) = |x|$ defined on $\mathbb{R}$, that the converse of Theorem 5.1 does not hold.

Question. Is there a function which is continuous everywhere on $\mathbb{R}$, but is not differentiable anywhere?

5.1 Algebra of derivatives

We now prove familiar rules of differentiation. You can already guess that many results for the *algebra of derivatives* will follow from the ________________________.

Theorem 5.2. Suppose that $f, g\colon (a, b) \to \mathbb{R}$ are differentiable at $c \in (a, b)$, we have:

(a) If $\alpha \in \mathbb{R}$, then $f + \alpha g$ is differentiable at c, and

(b) (____________________) The function fg is differentiable at c, and

(c) If $g(c) \neq 0$, the function $1/g$ is differentiable at c, and

(d) (____________________) If $g(c) \neq 0$, the function f/g is differentiable at c, and

The results of the algebra of derivatives can be extended to calculate the derivative of, say, the function $f_1 f_2 \dots f_n$ using ________________.

Proof of the algebra of derivatives:

5.2 Carathéodory's Theorem

Here is an interesting alternative criterion for differentiability. This theorem is named after the Greek mathematician, Constantin Carathéodory (1873-1950) who published influential works on analysis and mathematical physics.

Theorem 5.3. [*Carathéodory's Theorem*] Let $f\colon I \to \mathbb{R}$ be defined over an interval I, and let $c \in I$, then f is differentiable at c iff there exists a function $\varphi\colon I \to \mathbb{R}$ that is ________________ at c and satisfies the equation

$$= \qquad (*)$$

In addition, we have ______________________.

Proof:

Example 5. For any $n \in \mathbb{N}$, prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$ is differentiable at any $c \in \mathbb{R}$. Hence find $f'(c)$.

The previous example, combined with the algebra of derivatives, immediately gives us the following useful result.

Corollary. All polynomials are differentiable on $\mathbb{R}$.

Example 6. Use Carathéodory's Theorem to prove the Product Rule.

(Compare with our previous proof using limits, which proof do you prefer?)

Theorem 5.4. [__________________] Let $g\colon I \to \mathbb{R}$ and $f\colon J \to \mathbb{R}$, where I and J are intervals, and $f(J) \subseteq I$. Let $c \in J$. Suppose f is differentiable at c and g is differentiable at $d = f(c)$, then the composition $g \circ f$ is differentiable at c with

$$=$$

Proof: Study the proof in the Quiz (based on Carathéodory's Theorem) and write your own version on a separate page.

Theorem 5.5. [*Derivative of inverse functions*] Let $f: I \to \mathbb{R}$ be a continuous, strictly monotone function on an interval I. Let $f^{-1}: f(I) \to \mathbb{R}$ be the inverse function of f. If f is differentiable at c, and $f'(c) \neq 0$, then f^{-1} is differentiable at $d = f(c)$, with

$$(f^{-1})'(d) =$$

Proof:

Note: some books call the above Theorem the 'Inverse Function Theorem' (IFT). Indeed, in future analysis courses, you will study a more general version of the IFT (in $\mathbb{R}^n$) that resembles Theorem 5.5. But for now, in this course the IFT will refer to Theorem 4.5.

Example 7. Find the derivative of the inverse sine function.

Lemma 5.6. The function $f\colon (0,\infty) \to \mathbb{R}$, where

$$f(x) = x^r, \qquad \text{where } r \text{ is a positive rational number,} \tag{$\star$}$$

is differentiable at any $x > 0$, with $f'(x) = rx^{r-1}$.
[*Do this in 3 steps: (1)* $r = n \in \mathbb{N}$, *(2)* $r = 1/n$, $(n \in \mathbb{N})$, *(3)* $r = m/n$, $(m, n \in \mathbb{N})$.]

Reminder: We don't know what an irrational exponent like $x^{\sqrt{2}}$ means at this point!

Here are some miscellaneous definitions regarding the concept of differentiability.

(a) We have so far been discussing differentiability at a point. If a function f is differentiable at all points in an open interval (a, b) then we say that f is differentiable ____________.

(b) The definition of the derivative at the beginning of this chapter can be extended to the endpoints of a closed bounded interval. For example, the function $f : [a, b] \to \mathbb{R}$ is differentiable at a if

(c) If f is differentiable on (a, b) and f' is itself differentiable at $c \in (a, b)$, we can write the derivative of f' at c as

$$=$$

We call this the _________________________ of f at c.

(d) If a function is sufficiently 'nice', we can carry on differentiating to obtain the n-th derivative of f at c, denoted __________.

(e) A derivative of a function on an interval I need not be continuous on I, but it would be a nice bonus if it is!

If f' is continuous on I, f is said to be ______________________________.

If $f^{(n)}$ is continuous on I, f is said to be *n-times continuously differentiable.*

(f) The symbol __________ is used to denote the set of all functions that are n-times continuously differentiable on the interval I. For instance, the following expressions are all equivalent.

- f is continuously differentiable on I.
-
-

In this terminology, another way to say "*f is continuous on I*" is __________.

(g) Finally, the ultimate 'nice' function: If f is n-times continuously differentiable on I for all $n \in \mathbb{N}$, then we say that f is _______________, or $f \in$ __________.

Question. Think of a function defined on $\mathbb{R}$ which is a) C^1 but not C^2, b) C^∞.

5.3 Local-Extremum Theorem

The next application of the derivative is related to something you know from school: The derivative vanishes at 'turning points'. But how do we prove this without pictures?

Definition. Let I be an interval. The point $c \in I$ is said to be an *interior point* of I if

Definition. Let I be an interval and let $f\colon I \to \mathbb{R}$. Let c be an interior point of I.

- f has a *local minimum* at c if
- f has a *local maximum* at c if
- A local ______________ is either a local maximum or a local minimum.

Theorem 5.7. [*Local-extremum theorem*] Let c be an interior point of the interval I. Suppose $f\colon I \to \mathbb{R}$ has a local extremum at c. If $f'(c)$ exists, then $f'(c) = 0$.
Proof:

Corollary. Let c be an interior point of the interval I. Suppose $f\colon I \to \mathbb{R}$ has a local extremum at c, then either $f'(c) = 0$ or ____________________________.

5.4 The Mean-Value Theorem (MVT)

We now study an important application of the derivative. We will show that a vibrating string fixed at both ends attains an extremum somewhere. This theorem is named after *Michel Rolle* (1652-1719), a French self-taught mathematician who also published the technique known today as *Gaussian elimination*.

Theorem 5.8. [*Rolle's Theorem*] If $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), with $f(a) = f(b) = 0$, then

Proof: [*In some books, Rolle's Theorem only requires* $f(a) = f(b)$ *(possibly nonzero).*]

Theorem 5.9. [*Mean-Value Theorem (MVT)*] If $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ such that

$$= \qquad (*)$$

Proof:

Here are some interesting applications of the MVT (more in Assignment 4 and in the Quiz). How you would have proved the following Lemma back in school?

Lemma 5.10. Prove that if $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, and that $f'(x) = 0$ for all $x \in (a,b)$, then $f =$________________ on $[a,b]$.

Lemma 5.11. [________________________________] If $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then f is increasing on $[a,b]$ iff $f'(x) \geq 0$ for all $x \in (a,b)$

The results from the Lemma (5.11) need to be applied with extreme caution.

Example 8. True or false?

(a) Let $f\colon [a,b] \to \mathbb{R}$ be differentiable on the interval (a,b). If f is strictly increasing on $[a,b]$ then $f'(x) > 0$ for all $x \in (a,b)$.

(b) If f is differentiable at c and $f'(c) > 0$, then there exists a neighbourhood of c in which f is strictly increasing.

Example 9. Use the MVT to prove the inequality $\tan x > x$ for all $x \in (0, \pi/2)$. State all assumptions clearly.

Next, we prove a familiar result about using the second derivative to classify the nature of a local extremum. But we will need this:

Lemma 5.12. [*Sign-Preservation Lemma for limits*] Let $g : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. If $\lim_{x \to c} g(x) > 0$, then $\exists \delta > 0$ such that

Theorem 5.13. [______________________________] Let $f : (a, b) \to \mathbb{R}$ be differentiable in a neighbourhood of $c \in (a, b)$, with $f'(c) = 0$. Suppose that f' is differentiable at c.

(a) If $f''(c) < 0$ then f has a local _______________ at c.

(b) If $f''(c) > 0$ then f has a local _______________ at c,

Example 10. Does the converse to the above Theorem hold?

Here is an alternative version of the MVT, involving 2 functions at once, which will be useful in the next section. *Augustin-Louis Cauchy* (1789-1857) was a prolific French mathematician and one of the founding fathers of Analysis.

Theorem 5.14. [*Cauchy Mean-Value Theorem (CMVT)*]: Let f and g be continuous on $[a, b]$ and differentiable on (a, b), and assume that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that

$$=$$

(Note: By putting $g(x) = x$, we see that we recover the ___________.)
Proof: Define a function $h\colon [a, b] \to \mathbb{R}$ which is continuous on $[a, b]$ and differentiable on (a, b) as:

$$h(x) = f(x) - \lambda g(x),$$

where the constant λ is to be determined. As in the proof of the MVT, we require:

5.5 L'Hôpital's Rules

We can now expand our limit-calculation repertoire by combining limits with differentiation. In particular, we are interested in limits of the form $\lim_{x \to c} \frac{f(x)}{g(x)}$ where both f and g approach 0 or ∞. These theorems were discovered by *Johann Bernoulli* (1667-1748), a Swiss mathematician considered one of the founders of Calculus. Bernoulli was the tutor of the *Marquis de L'Hôpital,* who published a book based on Bernoulli's lessons.

5.5.1 Indeterminate forms

If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or ∞, then $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ is said to be in the *indeterminate form*:

An indeterminate form can turn out to be a real number or $\pm\infty$. Can you think of examples of $f(x)$ and $g(x)$ for these cases?

Very strong warning: The only time when you get to write these absolutely crazy notations is in connection with limits. Do not use these symbols in other contexts unless explicitly advised by your lecturers.

5.5.2 L'Hôpital's Rule 0

We now show that the limit of the form $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ is naturally connected to derivatives.

Theorem 5.15. [*L'Hôpital's Rule 0*]: Let f and g be defined on $I \supset [a, b]$. Let $f(a) = g(a) = 0$ and $g(x) \neq 0$ for $x \in (a, b)$. If f and g are differentiable at a with $g'(a) \neq 0$, then

$$=$$

Proof:

Example 11. Calculate $\lim_{x\to 0} \dfrac{\sin x}{x}$.

5.5.3 L'Hôpital's Rule 1

The next version allows us to calculate $\lim_{x\to a^+} \frac{f(x)}{g(x)}$ in the indeterminate form $\frac{0}{0}$.

Theorem 5.16. [*L'Hôpital's Rule 1*]: Let f and g be differentiable on (a,b) with $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose that $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$, then,

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \implies$$

where L may be a real number or $\pm\infty$.

Proof:

Note the logic of the theorem: If $\lim f'/g'$ exists, then it equals $\lim f/g$. However, in our working, I recommend the following steps:

Step 0) Define f and g and pick the interval on which $g'(x) \neq 0$.

Step 1) Check if $\lim(f/g)$ is in an indeterminate form.

Step 2) Try calculating $\lim(f'/g')$.

Step 3) If Step 2 gives L, then feed it back to $\lim(f/g)$ "*by L'Hôpital's Rule*".

Example 12. Calculate: a) $\lim\limits_{x\to 0^+} \dfrac{\sin x}{\sqrt{x}}$, b) $\lim\limits_{x\to 0^+} \dfrac{1-\cos x}{x^3}$.

5.5.4 L'Hôpital's Rule 2

Next we deal with the indeterminate form $\dfrac{\infty}{\infty}$.

Theorem 5.17. [*L'Hôpital's Rule 2*]: Let f and g be differentiable on (a,b) with $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose that $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = \pm\infty$, then,

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \implies$$

where L may be a real number or $\pm\infty$.

Proof: Choose any $x, y \in (a, b)$ such that $x < y$. We know that f and g are continuous on $[x, y]$ and differentiable on (x, y).
We also know that, $\forall t \in (x, y)$, $g'(t)$__________. By the __________, $\exists c \in (x, y)$ such that

$$\frac{f'(c)}{g'(c)} =$$

Now let's express $f(x)/g(x)$ as terms which we can control individually.

$$\frac{f(x)}{g(x)} =$$

Here are some extensions of our results.

- **Two-sided extension:** An almost identical working gives L'Hôpital's Rules 1,2 for left-handed limits. Combining the one-sided versions gives the two-sided version of L'Hôpital's Rule.

Theorem 5.18. [*Two-sided L'Hôpital's Rule*]:. Let f and g be differentiable on $(a, b) \setminus \{c\}$ with $g'(x) \neq 0$ for all $x \in (a, b) \setminus \{c\}$. Suppose that $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$, then,

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L \implies$$

where L may be a real number or $\pm\infty$.

- **Limit at ∞ extension:** Some adjustment of the proofs will allow us to establish L'Hôpital's Rules 1,2 for $\lim_{x \to \pm\infty} \frac{f'(x)}{g'(x)}$. [by working in the bounded interval $(\alpha, \beta) \subset (a, b)$.] This results in further variations of the one-sided L'Hôpital's Rules. For example, taking $b \to \infty$, we have

Theorem 5.19. [*L'Hôpital's Rule for limits at* ∞]:. Let f and g be differentiable on (a, ∞) with $g'(x) \neq 0$ for all $x \in (a, \infty)$. Suppose that $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$ or $\pm\infty$, then,

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L \implies$$

where L may be a real number or $\pm\infty$.

Example 13. Evaluate the following limits.
a) $\lim_{x \to \infty} \frac{x^2}{x^2 + 3x + 1}$ b) $\lim_{x \to \infty} \frac{x - \sin x}{2x + \sin x}$.

CHAPTER 6

TAYLOR'S THEOREM

You may be familiar with the approximation of a function $f(x)$ by a polynomial (so-called Taylor or Maclaurin series). You may remember this kind of series:

$$f(x) = f(x_0) + \qquad (x - x_0) + \qquad (x - x_0)^2 + \dots \qquad (x - x_0)^n + \dots$$

At university, we are not only concerned with what the series looks like, but also the error term (the '*remainder*') in the above expansion, and the convergence of the series (given what we know from Analysis I). We will be occupied with these questions for the rest of the course. Along the way, we will be reunited with our old friends: exp and log.

Theorem 6.1. [________________________] Let $I = [a, b]$ and $n = 0, 1, 2 \dots$. Suppose that $f \in C^n(I)$ and that f is $(n+1)$-times differentiable on (a, b). If $x_0 \in I$, then, $\forall x \in I \setminus \{x_0\}$, $\exists c$ between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n$$

where $R_n =$

Note that when $x = x_0$, the equality is trivial.

Although the theorem bears the name of English mathematician *Brook Taylor* (1685-1731) who studied the polynomial expansion, it was *Joseph-Louis Lagrange* (1736-1813) who provided the remainder term. R_n is often called the

Proof: We are given $x, x_0 \in I = [a, b]$ with $x \neq x_0$. let J denote the closed interval $[x, x_0]$ or $[x_0, x]$ (whichever makes sense). Clearly $J \subseteq I$. Define the function $P\colon J \to \mathbb{R}$ by:

$$P(t) = f(t) + (x - t)f'(t) + \frac{(x-t)^2}{2}f''(t) + \ldots + \frac{(x-t)^n}{n!}f^{(n)}(t).$$

Since $f \in C^n(J)$, P is continuous on J and differentiable on (x, x_0) or (x_0, x) [why?]. Observe that the values of P and f agree at the endpoint $t =$_____. The derivative $P'(t)$ is given by:

Let's define $R\colon J \to \mathbb{R}$ by

$$R(t) = f(x) - P(t).$$

We note that when $t =$ _____, we obtain the expansion in the Theorem, with the remainder term given by _______ (to be determined). Also, note that $R'(t) =$ __________

Define $G\colon J \to \mathbb{R}$ by:

$$G(t) = R(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} R(x_0).$$

Clearly, G is also continuous on J and differentiable on (x, x_0) or (x_0, x). The derivative $G'(t)$ is given by:

We claim that the remainder term can be obtained by applying Rolle's Theorem to G on J.

Example 1. Use Taylor's Theorem with $n = 2$ and $x_0 = 0$ to approximate $\sqrt[3]{1+x}$ where $x > -1$. Hence give a numerical approximation of $\sqrt[3]{1.3}$. How accurate is this estimate?

Example 2. Prove that $\cos x \geq 1 - \frac{1}{2}x^2$ for all $x \in \mathbb{R}$.

6.1 Taylor series

Definition. Let $f \in C^\infty([a,b])$. The *Taylor series* (or *Taylor expansion*) of f about a point $x_0 \in [a,b]$ is the infinite series

$$\sum$$

Warning: Beware of the difference between *Taylor series* and *Taylor's Theorem.*

We have not said that $f(x)$ equals the above series. We do not even know if the series converges. The convergence of Taylor series is a delicate question that we will keep investigating for the rest of the course. The series may converge for some values of x but not others. Even so, the series may converge to something that is not f!

Definition. We write

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k,$$

if and only if the sequence of remainder terms $R_n(x)$ converges to 0 for each x in the open interval (a,b).

Note: We usually test the convergence at the boundary points a, b separately.

Lemma 6.2. [____________________] Let (a_n) be a sequence such that $a_n > 0$. Suppose $0 < l < 1$ and $a_{n+1}/a_n \leq l$ eventually, then ________________.

Lemma 6.3. Let $I = (a,b)$ and suppose $f \in C^\infty(I)$. Suppose that there exists a constant M such that for all $x \in I$ and all $k \in \mathbb{N}$, $|f^{(k)}(x)| \leq M$, then the Taylor series of f about $x_0 \in I$ converges to $f(x)$ for all $x \in I$.

Corollary. $\forall x, x_0 \in \mathbb{R}$, the Taylor series for $\sin x$ and $\cos x$ converge to those functions.

It's worth remembering the expressions for these series around $x_0 = 0$.

6.2 The exponential function

Definition. The *exponential function*, $E \colon \mathbb{R} \to \mathbb{R}$, is a differentiable function such that

$$\forall x \in \mathbb{R}, \quad E'(x) = E(x), \quad \text{with } E(0) = 1.$$

Suppose for now that $E(x)$ exists. If so, $E \in C^\infty(\mathbb{R})$ by induction.Taylor's Theorem then tells us that, around $x_0 = 0$,

We now show that for each $x \in \mathbb{R}$, $R_n \to 0$ as $n \to \infty$.

Thus, for all $x \in \mathbb{R}$ the Taylor series of $E(x)$ converges to $E(x)$, and we can write

$$E(x) = \sum$$

We can't yet verify that $E(x)$ in this form satisfies $E'(x) = E(x)$ because we still don't know how to differentiate the series term by term. We'll come back to this point.

Here are some familiar properties of the exponential function that follow from the definition. Properties (a) and (f) are proved in the Quiz.

(a) E is unique.

(b) $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$.

(c) $E(x) > 0$ for all $x \in \mathbb{R}$

(d) E is strictly increasing on $\mathbb{R}$

(e) $\lim_{x\to\infty} E(x) = \infty$, $\lim_{x\to-\infty} E(x) = 0$, and $E(\mathbb{R}) = (0, \infty)$.

(f) $E(r) = e^r$ for all $r \in \mathbb{Q}$, where the constant $e \equiv E(1)$.

6.3 The logarithm

Since $E : \mathbb{R} \to E(\mathbb{R}) =$ _______ is continuous and strictly increasing on $\mathbb{R}$, the ________________________ Theorem tells us that

Definition. The logarithm function, $L\colon (0,\infty) \to \mathbb{R}$, (also called, *natural logarithm*, '*natural log*', or simply '*log*') is defined as the inverse function of the exponential function.

Here are some other familiar properties of the log that follow from this definition.

(a) $L(1) = 0$ and $L(e) = 1$.

(b) $L'(x) = 1/x$ for $x \in (0,\infty)$.

(c) $L(xy) = L(x) + L(y)$, $\forall x, y \in (0,\infty)$.

(d) $L(x^r) = rL(x)$ for all $x > 0$ and $r \in \mathbb{Q}$.

(e) $\lim_{x\to\infty} L(x) = \infty$ and $\lim_{x\to 0^+} L(x) = -\infty$.

Definition. [*The power function*] If $\alpha \in \mathbb{R}$ and $x > 0$, the number x^α is defined to be

$$x^\alpha =$$

The power function allows us to extend property (f) of the exponential, and we can now write $E(x)$ as e^x for any $x \in \mathbb{R}$. Similarly, property (d) of the log now holds for all real exponents. With this definition, it is easy show that x^α can be differentiated using the same rule as in Lemma 5.6.

New notation. From now on we will denote the exponential and the log as follows:

$$E(x) \equiv e^x, \qquad\qquad L(x) \equiv \ln x.$$

Example 3. Apply Taylor's Theorem to $f(x) = \ln(1 + x)$ around $x_0 = 0$. Show that the Taylor series converges[1] to f when $x \in (0, 1)$. What happens at $x = 1$?

[1] In fact, the the Taylor series also converges to $\ln(1 + x)$ when $-1 < x < 0$, but our proof here won't work (why?). What is needed is a different technique (for example, integrating another Taylor series, which you will get to do in the next analysis courses).

6.4 Interlude: limsup and liminf

Consider the sequence $x_n = (-1)^n$. Clearly $\lim_{n\to\infty} x_n$ does not exist. However, consider the sequence

$$S_n = \sup_{i\geq n} x_i.$$

We note that S_n converges with $\lim_{n\to\infty} S_n =$ _____. Similarly, we could define:

$$I_n = \inf_{i\geq n} x_i =$$

$$\lim_{n\to\infty} I_n =$$

Definition. Let (x_n) be a sequence of real numbers.

(a) The ________________________ of (x_n), denoted $\limsup_{n\to\infty} x_n$ or $\overline{\lim}_{n\to\infty} x_n$ is defined as

(b) The ________________________ of (x_n), denoted $\liminf_{n\to\infty} x_n$ or $\underline{\lim}_{n\to\infty} x_n$ is defined as

Example 4. Write down limsup and liminf for the following sequences:

$$x_n = \begin{cases} 5^{-n} & \text{if } n \text{ is odd} \\ -1/n & \text{if } n \text{ is even} \end{cases} \qquad y_n = \begin{cases} 2^n & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Here are some observations about limsup and liminf (proofs needed).

- If (x_n) is bounded, (S_n) is ____________________ sequence.
- If $\lim_{n\to\infty} x_n = \infty$ then $\limsup_{n\to\infty} x_n =$ _____.
- $\lim_{n\to\infty} x_n = L < \infty \iff \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n =$ _____ $< \infty$.

Here is a useful series-convergence test to add to your list of tests from Analysis I.

Theorem 6.4. [*Cauchy's Root Test*] Consider the series $\sum_{n=1}^{\infty} x_n$. Let $r = \limsup_{n\to\infty} |x_n|^{1/n}$.

(a) If $r < 1$, then $\sum_{n=1}^{\infty} x_n$ is convergent,

(b) If $r > 1$, then $\sum_{n=1}^{\infty} x_n$ is divergent.

6.5 Radius of convergence

Definition. A power series around $x = x_0$ is the infinite series of the form $\sum_{n=N_0}$

We are interested in the question of convergence of power series, so the value of N_0 is irrelevant (sometimes we will simply denote the series as $\sum a_n x^n$). Also, WLOG we can concentrate on the power series around $x_0 = 0$.

Analysis I has already provided us with several 'tests' to determine whether a series converges or not. Please revise them. Here is an old friend.

Theorem 6.5. [*Ratio Test*] Suppose $T_n \neq 0$ eventually, and $= L$. If $L < 1$, then $\sum T_n$ converges. If $L > 1$, then $\sum T_n$ diverges.

Example 5. Apply the Ratio Test to the following series and deduce the values of x for which each series converges or diverges.

a) $\displaystyle\sum_{n=0}^{\infty} n!x^n$ b) $\displaystyle\sum_{n=0}^{\infty} (n-10)x^n$ c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Convergence tests tell us that about the set A such that "*if $x \in A$, then the series converges*". But the tests do not give us *all* the values of x for which a series converges, as the next example shows. Convergence at the 'boundary' needs to be tested separately.

Example 6. Applying the Ratio Test to the following series shows that they converge if __________ and diverge if __________. What happens at $x = \pm 1$?

a) $\displaystyle\sum_{n=0}^{\infty} x^n$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

Definition. Suppose the power series $\sum a_n x^n$ converges if $|x| < R$ and diverges if $|x| > R$. We call R the ________________________________ of the power series.

6.6 Cauchy-Hadamard Theorem

The next theorem is one the most powerful tools with which we can study the convergence of power series. It is named after our old friend Cauchy and the French mathematician *Jacques Hadamard* (1865-1963), whom you may hear about again when you study the *Prime-Number Theorem.*

Theorem 6.6. [*Cauchy-Hadamard Theorem*] Let $\sum a_n x^n$ be a power series.

Let $\rho =$ ________________________ and define $R = \begin{cases} 1/\rho & \text{if } 0 < \rho < \infty, \\ 0 & \text{if } \rho = \infty, \\ \infty & \text{if } \rho = 0. \end{cases}$

(a) If $0 < R < \infty$, then the series converges whenever $|x| < R$, and diverges whenever $|x| > R$.

(b) If $R = 0$, then the series converges only for $x = 0$.

(c) If $R = \infty$, then the series converges for all $x \in \mathbb{R}$.

Some remarks about the Cauchy-Hadamard Theorem.

- The theorem tell us that the subset of $\mathbb{R}$ on which a power series converges is an ____________ and not any random, fragmented subset.

- The convergence at $x = \pm R$ must be tested separately. This explains why, in §6.1, we studied the convergence of Taylor's series defined on an open interval (a, b), and tested the convergence at the endpoints separately.

- The Root Test (and hence the Cauchy-Hadamard Theorem) implies the Ratio Test (in other words, the Ratio Test is a weaker test[2]). But it may sometimes be easier to obtain R from the Ratio Test as we did in Examples 5 and 6. We tend to invoke the Cauchy-Hadamard theorem when a_n contains n-th powers, or when a_n takes different forms depending on, say, when n is odd or even.

Example 7. Find the radius of convergence of the series $\sum_{n=1}^{\infty} a_n x^n$ where:

a) $a_n = 3^n$ b) $a_n = \begin{cases} 2^n & n \text{ odd} \\ 5^n & n \text{ even} \end{cases}$ c) $a_n = \dfrac{1}{n^n}$.

[2] See `https://tinyurl.com/4t5v8wut` (not examinable).

6.7 Term-by-term differentiation

Can power series be differentiated? This is not obvious because we are dealing with an infinite sum. *If* we could, then that would settle the question of the existence of the exponential, which we proposed to be the Taylor series

$$e^x =$$

If permitted, then the differentiated power series looks like:

which is exactly e^x as we had hoped! The ability to differentiate power series is extremely useful in (real and complex) analysis, and it turns out that indeed we *can* differentiate, as long as we stay inside the radius of convergence.

Theorem 6.7. [*Term-by-Term differentiation*] A series $\sum a_n x^n$ can be differentiated term-by-term within its radius of convergence.

We will break up the proof of this epic final theorem into bite-size lemmas.

Lemma 6.8. (a) [*Pulling a limit out of a limsup.*] If $\lim_{n\to\infty} x_n = x > 0$ and $\limsup_{n\to\infty} y_n = y$, then $\limsup_{n\to\infty} (x_n y_n) = xy$.

(b) The series $\sum_{n=1}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence.

We could repeat the previous proof and obtain the following

Corollary. The following series have the same radius of convergence:

$$\sum_{n=0}^{\infty} a_n x^n \qquad \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \sum$$

Proof of Theorem 6.7: Let $f\colon (-R,R) \to \mathbb{R}$ and $g : (-R,R) \to \mathbb{R}$ be defined by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

By the previous Corollary, the radii of convergence of both f and g are equal. Call this radius R.

We want to show that, for all $x_0 \in (-R,R)$, $f'(x_0) =$ __________. In terms of limits, we want to show that

Or, equivalently, that:

The terms in the infinite sums could be rearranged because, within the interval of convergence $(-R,R)$, the series are ____________________ convergent, as shown in the proof of Theorem 6.4. (You may need to look up *series rearrangement* and Riemann rearrangement theorem from *Analysis I*.)

Lemma 6.9. Let $x_0 \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for any $x \in \mathbb{R}$ with $x \neq x_0$, we have the estimate

$$\left| \frac{x^n - x_0^n}{x - x_0} - n x_0^{n-1} \right| \leq \frac{n}{2}(n-1)\rho^{n-2}|x - x_0|,$$

where $\rho = \max(|x_0|, |x|)$.

Proof: Use Taylor's Theorem to find a linear approximation $f(x) = x^n$ around $x = x_0$.

Proof of Theorem 6.7 continued: So far we now have the estimate

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| \leq \left[\sum_{n=2}^{\infty} |a_n| \frac{n}{2}(n-1)\rho^{n-2} \right] |x - x_0|,$$

where $\rho = \max(|x_0|, |x|)$.

From the proof of Cauchy's Root Test, we showed that a series converges _______________ within its radius of convergence. This means that series $\sum a_n x^n$, $\sum |a_n x^n|$ and $\sum |a_n n(n-1)x^{n-2}|$ all converge. And since $\rho < R$, the series in the square brackets above also converges to a finite number, say, A_x.

Now take the limit as $x \to x_0$ on both sides. Note that A_x converges to a finite number. We then have:

□

Here are some immediate corollaries of the term-by-term differentiation theorem.

(a) The power series $\sum a_n x^n$ is continuous on $(-R, R)$.

(b) The power series $\sum a_n x^n$ is infinitely differentiable in $(-R, R)$.

These results are very handy when we define functions like the exponential as a power series - we get continuity and differentiability for free within the interval of convergence.

Finally, we have a lemma which allows us to obtain new power series by differentiation.

Lemma 6.10. [*Uniqueness of power series*]
Suppose that, for all $x \in (-R, R)$, we have two convergent power series such that $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$, then

[*In other words, we can compare coefficients of two power series term-by-term.*]

Example 8. Starting from a geometric series, obtain the Taylor series about $x_0 = 0$ for $\frac{1}{(1-x)^2}$. Find the radius of convergence.

Example 9. In this course we have shown that if the sine function is defined geometrically (via angles), then it is C^∞ on $\mathbb{R}$, and its Taylor series is given by

$$\sin x =$$

(a) Show that we can take the reverse perspective, *i.e.* if we *define* the sine function by the power series above, show that it is C^∞ on $\mathbb{R}$.

(b) Define the cosine function as a series and verify that $(\sin)' = \cos$ and $(\cos)' = -\sin$

(c) Prove that $\sin^2 x + \cos^2 x = 1$ for all $x \in \mathbb{R}$.