differential geometry

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These notes accompany the course *32330: Differential Geometry* for third-year undergraduates reading mathematics at the University of Hull.

We will delve into fundamental properties of curves and surfaces, largely in 3D. Many results from Vector Calculus, Analysis and Linear Algebra will be assumed.

The main textbook for this course is

Elementary Differential Geometry, Pressley A., 2^{nd} ed., Springer (2010).

This is available as an e-book from BJL. In addition, I recommend the following books, also in BJL.

Differential Geometry of Curves and Surfaces, do Carmo M. P., Prentice-Hall (1976).

Elementary Differential Geometry, Bär C., Cambridge University Press (2010).

Differential Geometry of Curves and Surfaces, Banchoff T. and Lovett S., 2^{nd} ed., CRC Press (2015).

Although MATLAB is not required for this course, you might find it helpful for plotting curves and surfaces that would otherwise be difficult to visualise. If you don't have MATLAB on your computer – send an email to `help@hull.ac.uk` to request the installation link.

Please send comments, questions and corrections to `s.chongchitnan@hull.ac.uk`.

SC
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How to plot 3D curves and surfaces in MATLAB?

a) Curves

A good starting point is to read the help file – search for "*Common Graphics Functions*".

Here's a sample code for plotting the 3D curve

$$\boldsymbol{\gamma}(t) = (\cos t, \sin t, t), \qquad 0 < t < 6\pi.$$

in MATLAB. Write a little M-file and save it for future use.

```
syms t
x = cos(t);
y = sin(t);
z = t;
p = ezplot3(x, y, z, [0,6*pi]);
p.LineWidth=3;     % these two lines are...
p.Color='blue';    % ... optional.
```

The main command is `ezplot3` which uses 300 points by default. If you require a better resolution then you should look up the `plot3` command. If you just want a 2D parametric plot, use `ezplot`.

You can achieve the same thing in MuPAD which has a slightly nicer interface.

```
1 x := cos(t):
2 y := sin(t):
3 z := t:
4 p := plot::Curve3d(
5 [x, y, z], t = 0..6*PI,
6 AdaptiveMesh = 1,
7 LineColor = RGB::Blue,
8 LineWidth = 0.7
9 ):
10 plot(p)
```

Lines 6, 7 and 8 are optional.

The option '`AdaptiveMesh`' invokes adaptive resolution (*i.e.* more points around interesting bits of the graph). It can take a value 1 2 or 3, giving you finer and finer resolution (but a very fine plot will take longer to plot and more difficult to manipulate with the mouse). The default LineWidth is 0.35. Look up `Curve3d` for more options.

If you just want a 2D parametric plot, use `Curve2d`.

b) Surfaces

Here are some sample codes for plotting a 3D surface parametrized by

$$\boldsymbol{\sigma}(u,v) = (u\cos(v), u\sin(v), u), \qquad 0 < u < 1,\ 0 < v < 2\pi.$$

In MATLAB, I recommend the `ezsurf` command.

```
syms u v
x = u*cos(v);
y = u*sin(v);
z = u;
ezsurf(x, y, z, [0,1, 0,2*pi])
alpha(0.2)            % Optional. This makes the plot a bit transparent.
```

The default resolution of `ezsurf` is 60 points in each of the u and v interval. Note that surfaces of the form $z = f(x,y)$ can be plotted this way by letting $x = u$ and $y = v$.

In MuPAD, use the `plot::Surface` command.

```
1 x := u*cos(v):
2 y := u*sin(v):
3 z := u:
4 p := plot::Surface([x, y, z], u = 0..1, v = 0..2*PI,
5 AdaptiveMesh=1,
6 FillColor = RGB::Blue.[0.4],
7 FillColor2 = RGB::Pink.[0.4]
8 ):
9 plot(p)
```

Lines 5, 6 and 7 are optional. The number 0.4 makes the surface a bit transparent.

Chapter 1

Curves

Our starting point is curves in 2D and 3D, described in the language of *vectors*.

1.1 Parametrized curves

Definition. A *parametrized curve* in $\mathbb{R}^n$ is a vector-valued function

$$= \tag{1.1}$$

where each component $\gamma_i(t)$ is a real function, and ______________ (some subset of $\mathbb{R}$).

I will use a bold $\boldsymbol{\gamma}$ to denote a curve (vector). Unless stated otherwise, we will assume that all curves dealt with in this course can be differentiated *infinitely many times*, in other words, each γ_i is a _____ function.

It gets tiring to say "$\boldsymbol{\gamma}$ is infinitely differentiable", so instead, we say "$\boldsymbol{\gamma}$ is ______________". Normally, smooth means C^1, but only in this course do we take this word to mean C^∞.

Even if a function looks a bit suspicious, *e.g.* $\boldsymbol{\gamma}(t) = \big(\ln(t), \sqrt{t}\big)$, we will work with the interval of t for which $\boldsymbol{\gamma}$ is smooth (in this case, $I =$ __________).

Example 1. Write down the parametric equation of a line-segment joining two points with position vectors $\mathbf{a}$ and $\mathbf{b}$.

Example 2. (a) Find a vector parametrization of the curve $y = x^2$.

(b) Is $\boldsymbol{\gamma} = (t^2, t^4)$, $t \in \mathbb{R}$, a good parametrization?
If not, write down a better parametrization.

Example 3. Write down a vector parametrization of
a) the unit circle, b) the upper semi-circle.

Example 4. Sketch the 3D curve $\boldsymbol{\gamma}(t) = (\cos t, \sin t, t)$, $t \in (0, 4\pi)$.

Example 5. Find the Cartesian equation $\boldsymbol{\gamma}(t) = (e^t + 1, t^2)$, $t \in \mathbb{R}$. Sketch the curve.

Next we have a familiar result from Vector Calculus (consult your old notes).

Proposition 1.1. The tangent vector to the curve $\boldsymbol{\gamma}(t)$ is __________.

Example 6. Find the tangent vector to the curve in Example 5 at $t = 0$.

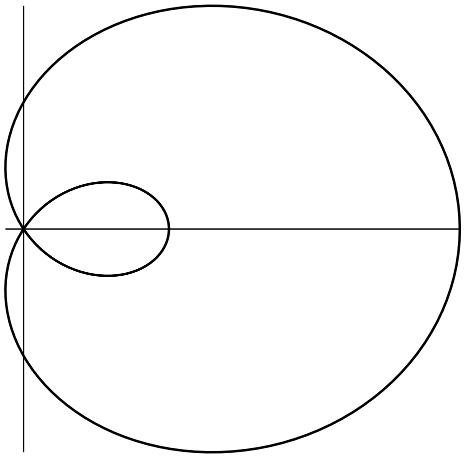
Example 7. A curve $\boldsymbol{\gamma}(t)$ has a constant tangent vector for all t. What curve is this? (PS. vector functions can be integrated and differentiated as usual – component-wise.)

Parametric curves can often intersect itself. This means that a single point on the curve can be specified by more than one value of t.

Example 8. Find the values of t for which the following curve[1] passes through the origin.

$$\boldsymbol{\gamma}(t) = \begin{pmatrix} (1+2\cos t)\cos t \\ (1+2\cos t)\sin t \end{pmatrix}, t \in [0, 2\pi].$$

Find the tangent vectors at those values of t.



[1]This is the *Limaçon of Pascal*. It is the locus traced out by a point on a circle rolling on another circle. See some nice graphics here: `http://bit.ly/1MZqifR`

1.2 Arc-length parametrization

Proposition 1.2. The *arc-length* of the curve $\boldsymbol{\gamma}(t)$ between $t = t_0$ and $t = t_1$ is given by

$$= \qquad (1.2)$$

See the Vector Calculus for proof, but we can roughly see why this is true as follows:

Definition. The arc-length *function* is defined as

$$= \qquad \text{for some constant } t_0. \qquad (1.3)$$

Important: (1.2) is a number, but (1.3) is a function, a very important function.

Definition. The scalar function $|\dot{\boldsymbol{\gamma}}(t)|$ is called the _______________ of the curve $\boldsymbol{\gamma}(t)$.

If we think about $\boldsymbol{\gamma}(t)$ physically as the trajectory (displacement) of a particle. Then the vector $\dot{\boldsymbol{\gamma}}(t)$ is just its _______________, and the magnitude $|\dot{\boldsymbol{\gamma}}(t)|$ is its _______________. The vector $\ddot{\boldsymbol{\gamma}}(t)$ is its ____________________.

Example 9. (a) Calculate the length of the helix which winds round a cylinder of unit radius *once*.

(b) Obtain the arc-length function, $s(t)$, with $t = 0$ as the initial point.

(c) Write down the parametrization of the helix using s instead of t. Find the speed of the curve in this parametrization.

Definition. The *arc-length parametrization* (also known as *unit-speed/ intrinsic/natural* parametrization) of $\boldsymbol{\gamma}(t)$ is $\tilde{\boldsymbol{\gamma}}(s)$, where s is the ______________________________ with an appropriately chosen starting point t_0.

Lemma 1.3. A curve parametrized by arc-length has unit speed.
Proof:

(We'll deal with the converse later.) It's important to keep in mind that both $\boldsymbol{\gamma}(t)$ and $\tilde{\boldsymbol{\gamma}}(s)$ describe the *same* curve - just parametrized differently. Where there is no confusion, we can write $\boldsymbol{\gamma}(s)$ instead of $\tilde{\boldsymbol{\gamma}}(s)$.

Example 10. Find the arc-length parametrization of the semicircle

$$\{(x, y) : x^2 + y^2 = R^2,\ x > 0\}.$$

Example 11. Find the arc-length parametrization of the 3D curve:

$$\boldsymbol{\gamma}(t) = (t, t^2, t^3),\ t > 0.$$

The moral of the story: it's sometimes not possible to find the arc-length parametrization explicitly.

But wait, why do we bother with arc-length parametrization in the first place? Later on, we will see that it simplifies many otherwise nasty formulae and proofs. Most of the time we won't need to know the explicit form of the arc-length parametrization, $s(t)$, but just the fact that it *exists* is enough.

1.3 Reparametrization

Let's generalise the idea of using a different parameter to describe the curve $\boldsymbol{\gamma}(t)$ as $\tilde{\boldsymbol{\gamma}}(s)$, where s is not necessarily the arc-length parameter.

Definition. A curve $\tilde{\boldsymbol{\gamma}}(s)$ is said to be a ____________________ of $\boldsymbol{\gamma}(t)$ if there is a bijection ϕ such that $\phi : S \to T$ (S and T are some subsets of $\mathbb{R}$) with

$$= \tag{1.4}$$

and ϕ and ϕ^{-1} are both *smooth*.

Important: the two curves are exactly the same (same image, same geometry) – only the points on it are described differently. The mapping on the right explains this situation. This kind of diagram will be very useful throughout the course.

Not any random parametrization is a *reparametrization*: we must check that ϕ is bijective and smooth, with smooth inverse,

Example 12. Let $\boldsymbol{\gamma}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, $t \in (0, \pi)$. Show that $\tilde{\boldsymbol{\gamma}}(s) = \begin{pmatrix} \sin s \\ \cos s \end{pmatrix}$ is a reparametrization of $\boldsymbol{\gamma}(t)$, and state the range of s.

Important: The function $t = \phi(s)$ can also be written (in short) as __________.
(Analogy: $y = f(x)$ can be written as $y(x)$ on a lazy day.)
Similarly, $s = \phi^{-1}(t)$ can be written as __________.
Similarly for derivatives: we can either deal with $\dot{\phi}(s) = d\phi/ds$ or __________.
(Analogy: $f'(x)$, df/dx and dy/dx all refer to the same thing). Be flexible.

Lemma 1.4. Let ϕ be a smooth bijection. Then the derivatives of ϕ and ϕ^{-1} are nonzero.
Proof:

Definitions. (a) A point on the curve $\boldsymbol{\gamma}(t)$ is a *regular point* if _______________, otherwise, it is a _________________ point.

(b) The curve $\boldsymbol{\gamma}(t)$ is regular if

Question. Find a regular and a non-regular parametrization for the line $y = x$.

This means that "regularity" is a property of parametrizations - and not the geometry of a curve. You can't tell whether a curve is regular by just looking at its shape.

Lemma 1.5. A reparametrization of a regular curve is again regular.
Proof:

The next theorem requires a little result from Analysis.

Lemma 1.6. Let f be a strictly increasing, smooth function in $[a, b]$, then f^{-1} exists, is smooth, and increasing on $[f(a), f(b)]$.

See any textbook on real analysis for a simple proof. Actually, only the fact that $f' \neq 0$ is enough for f to be invertible – this is the *inverse function theorem*. More about this later, but it's worth looking it up if you haven't heard of it before.

Theorem 1.7. A curve has a unit-speed reparametrization *iff* it is regular.
Proof: We will prove the '*if*' part (the '*only if*' part is homework).

$\square$

We've just shown the unit-speed reparametrization *exists* for all regular curves. But is it the arc-length?

Corollary. Let $\boldsymbol{\gamma}(u)$ and $\tilde{\boldsymbol{\gamma}}(s)$ be two unit-speed reparametrizations of a curve, then

$$u = \pm s + C, \tag{1.5}$$

where C is a constant.
Proof:

The previous Corollary implies that all unit-speed parametrizations are *effectively* the arc-length parametrization, but possibly with

(a) a shift in the parameter values. (b) increasing or decreasing parameters.

(or a combination of both. More about (b) in sheet 1, Q4). Let's think about this pictorially. Suppose we have a curve of length 10 units, what are some possible unit-speed parametrizations?

Whatever unit-speed parametrization you choose, it's always describing the same curve with length 10 units.

So from now on, we can use the terms "*arc-length parametrization*" and "*unit-speed parametrization*" interchangeably, keeping in mind these degrees of freedom.

In the next Chapter, we look at further properties of curves, including

- Curvature
- Torsion
- The Frenet Frame

I recommend revisiting Chapter 1 of your Vector Calculus notes.

Example 13. (a) Find an open interval I on which the curve

$$\boldsymbol{\gamma}(t) = (\cos^2 t, \sin^2 t), \quad t \in I,$$

is a regular curve.

(b) Find a unit-speed parametrization of the curve.

(c) Sketch the curve.

Chapter 2

Curvature and Torsion

In this Chapter, we will focus on two important characteristics of 3D curves: the curvature, which quantifies how much a curve curves, and the torsion, which measures the deviation of a curve from a 2D plane.

2.1 Curvature

So *how much does a curve curve?* A huge circle seems to curve a lot less than a little one (the outer lane of a big running track is mostly straight). We want to come up with a quantity which measures this *curvature*: let's call it call it κ. If R is the radius of a circle, then already we should expect

$$\propto \tag{2.1}$$

Definition. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve. The *curvature* is defined as the function

$$=$$

Important: The curvature is a function (scalar) – not a vector. Also, this definition only works for unit-speed parametrizations - we will deal with the general case later.

Now let's see if this definition of curvature is consistent with what we expect.

Example 1. Calculate the curvature of a circle centred at the origin with radius R.

Example 2. Prove that two unit-speed parametrizations of a curve have the same curvature.

This isn't surprising: the geometry of a curve is unchanged by reparametrizations.

Before we proceed, here are some useful results from vector calculus.

Warning: *These won't be given in the exam.*

(a) $\mathbf{a} \cdot \mathbf{b} =$

(b) $|\mathbf{a} \times \mathbf{b}| =$

(c) $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} =$

(d) $\mathbf{a} \cdot \mathbf{a} \times \mathbf{c} =$

(e) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) =$

(f) $|\mathbf{a} \times (\mathbf{a} \times \mathbf{b})| =$

(g) $\dfrac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) =$

(h) $\dfrac{d}{dt}(\mathbf{a} \times \mathbf{b}) =$

Theorem 2.1. The curvature of a regular curve, $\boldsymbol{\gamma}(t)$, is given by

$$\kappa(t) = \frac{|\boldsymbol{\gamma}'' \times \boldsymbol{\gamma}'|}{|\boldsymbol{\gamma}'|^3},$$

where a dash indicates a derivative wrt t.

Example 3. What curve is this?

$$\boldsymbol{\gamma}(t) = (a\cos t, a\sin t, bt),\ \ t \in \mathbb{R}. \tag{2.2}$$

Calculate its curvature. Discuss the cases when i) $a = 0$ $(b \neq 0)$, ii) $b = 0$ $(a \neq 0)$.

2.2 Curvature in 2D

In this section, let's assume that the curves are 2 dimensional, *i.e.* $\boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t))$. Whilst $\kappa \geq 0$, in 2D we should be able to say more about the *sign* of the curvature (*e.g.* $y = x^2$ VS $y = -x^2$).

Definition. The *unit tangent* to the curve $\boldsymbol{\gamma}(t)$ is defined as the vector

$$\mathbf{t} =$$

If $\boldsymbol{\gamma}(s)$ is a unit-speed curve, then clearly $\mathbf{t} =$ ________.

Definition. The *signed unit normal*, $\mathbf{n}_s$, is defined as the vector obtained by rotating the unit tangent by 90° anticlockwise, *i.e.* if $\mathbf{t} = (t_1, t_2)$, then

$$\mathbf{n}_s =$$

Note that $\mathbf{t} \cdot \mathbf{n}_s =$ _____.

Lemma 2.2. If $\mathbf{u}(s) \in \mathbb{R}^2$ with $|\dot{\mathbf{u}}| = 1$, prove that $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ are perpendicular.

Lemma 2.2, though very simple, will prove very useful for the rest of this Chapter.

Question. Why does Lemma 2.2 imply that $\ddot{\boldsymbol{\gamma}}$ and $\mathbf{n}_s$ are multiple of each other?

Definition. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve. The *signed curvature* is the scalar κ_s such that

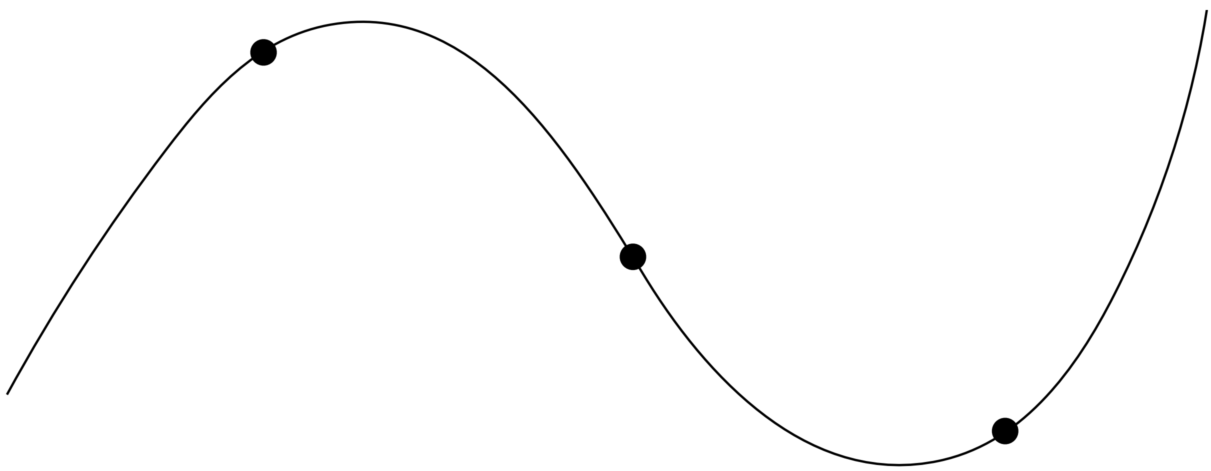
$$\ddot{\boldsymbol{\gamma}} = \kappa_s \mathbf{n}_s. \tag{2.3}$$

Whereas the curvature κ is always non-negative, κ_s can change sign. They have the same magnitude, as can be seen from the relation

$$\kappa = |\ddot{\boldsymbol{\gamma}}| = |\kappa_s \mathbf{n}_s| = |\kappa_s|.$$

Now we will argue that the concept of positive/negative curvature in 2D coincides with the 'concave up/down' concept from school maths.

Study the diagram of a 2D curve $\boldsymbol{\gamma}(s)$ below. Assume that the curve is swept out from left to right as s increases from 0. Mark in the directions of the unit tangents, signed normals, and $\ddot{\boldsymbol{\gamma}}$ at the marked points.
Note: It might help to think of the derivative of a vector as the change in the vector as it moves to a *nearby* point.



You can now see that the point where $\kappa_s > 0$ (*i.e.* where $\mathbf{n}_s$ and $\ddot{\boldsymbol{\gamma}}$ point in the same direction) is concave up (also called positive *concavity*), and similar for negative curvature/concavity. At $\kappa_s = 0$ we have a point of ______________, around which $\mathbf{t}$ is roughly constant.

So we see that the signed curvature is simply the generalisation of concavity.

The next Example shows what happens to Eq. 2.3 when $\boldsymbol{\gamma}$ is not unit speed.

Example 4. For a general curve $\boldsymbol{\gamma}(t)$ (not necessarily unit speed), show that

$$\mathbf{t}' = \kappa_s |\boldsymbol{\gamma}'| \mathbf{n}_s.$$

When we discuss curves with parameter s or t, it helps to distinguish between the two derivatives. I often use a dash for t and a dot for s derivative, but be flexible.

There is an intuitive parametrization of this rate of change, in terms of the *rotation* of the tangent vector $\mathbf{t}$.

Definition. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve in 2D. The *turning angle*, φ, is the angle that the tangent vector makes relative to the x axis, *i.e.*

$$\mathbf{t} = \dot{\boldsymbol{\gamma}}(s) = \tag{2.4}$$

Example 5. Roughly sketch the graph of $\varphi(s)$ for the curve segment on the previous page. Is this graph unique?

We can regard φ as a reparametrization $s \to \varphi(s)$. Indeed it's not too difficult to show that the turning-angle function, $\varphi(s)$, such as the one you sketched, exists for any unit-speed parametrized curve. Furthermore, $\varphi(s)$ will be smooth and unique (up multiples of 2π).[1]

[1]For an easy proof, see Bär, p. 37.

We now show that in fact the rate of change of the turning angle is exactly the signed curvature.

Lemma 2.3. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve in 2D and let $\varphi(s)$ be its turning angle. Then,

$$\kappa_s = \frac{\mathrm{d}\varphi}{\mathrm{d}s}. \tag{2.5}$$

Example 6. Find the arc-length function $s(t)$ for the curve $\boldsymbol{\gamma}(t) = (t, \cosh t)$ $(t > 0)$. Hence, use (2.5) to calculate $\kappa_s(s)$ and $\kappa_s(t)$.

It would have been more tedious to calculate from the definition (Eq. 2.3). See Problem sheet 2 for yet another way to do calculate κ_s.

Before we move on to curvature in 3D, we should mention a very special theorem: It turns out that the signed curvature characterises plane curves uniquely (up to rotation and translation). See Pressley for proof.

Theorem 2.4. [Fundamental Theorem of Plane Curves] Let κ_s: $I \to \mathbb{R}$ be any given smooth function ($I \subseteq \mathbb{R}$).

[Existence] There exists a unit-speed curve $\boldsymbol{\gamma} : I \to \mathbb{R}^2$ whose signed curvature is κ_s.

[Uniqueness] If $\tilde{\boldsymbol{\gamma}}$ is another unit-speed curve with signed curvature κ_s, then $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are related by a sequence of rotations and translations in $\mathbb{R}^2$.

2.3 Curvature in 3D

We've just shown that In 2D, the signed curvature κ_s essentially determines the shape of the curve. In 3D, however, another property is required to quantify how much a curve is 'lifted' out of the 2D plane: this is the *torsion*, as we will see shortly.

Definition. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve in $\mathbb{R}^3$, and let $\mathbf{t} = \dot{\boldsymbol{\gamma}}$ be the unit tangent. If the curvature $\kappa(s)$ is nonzero, we define the ______________________ as the vector

$$\mathbf{n}(s) = \frac{1}{\kappa(s)}\dot{\mathbf{t}}(s). \tag{2.6}$$

Example 7. Prove that:

(a) $\mathbf{n}$ is a unit vector.

(b) $\mathbf{n}$ is perpendicular to $\mathbf{t}$.

Definition. The *binormal* of a unit-speed curve is defined as the vector

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}. \tag{2.7}$$

Important: The order matters!

Clearly $\mathbf{b}$ is perpendicular to both $\mathbf{t}$ and $\mathbf{n}$, so $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ forms an ________________________.

Example 8. Find $|\mathbf{b}|$ and $\mathbf{n} \times \mathbf{b}$.

Thus we can perform calculations with $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ in the same way as we do with $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

Definition. The basis $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is called the *Frenet*[2] *frame* for 3D curves.

Question. What is the equivalent of the $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ frame for plane curves?

[2] *Jean Frédéric Frenet* (1816-1900), French mathematician. Sometimes also attributed to *Joseph Serret* (1819-1885), another French mathematician who independently studied 3D curves in this formulation.

Example 9. Prove that $\dot{\mathbf{b}}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{t}$.

This Example showed that $\dot{\mathbf{b}}$ is parallel to _____. Let $-\tau$ be the proportionality 'constant':

$$\dot{\mathbf{b}} = -\tau\mathbf{n}. \tag{2.8}$$

(the minus sign is just a convention which reduces the number of minus signs in our working later).

Now let's think about what τ means. If $\tau = 0$, then $\mathbf{b}$ is a constant vector. This means that the curve stays in the ______________ spanned by $\mathbf{t}$ and $\mathbf{n}$. We will prove this properly later, but that's the idea.

Definition. The *torsion* of a unit-speed curve with nonzero curvature is τ defined in Eq. 2.8.

Example 10. Prove that two unit-speed parametrizations of a curve $\boldsymbol{\gamma}(t)$ have the same torsion.

Solution: Let $\boldsymbol{\gamma}(s)$ and $\boldsymbol{\gamma}(u)$ be two unit-speed parametrizations of $\boldsymbol{\gamma}(t)$. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the basis calculated wrt to s, and $\{\tilde{\mathbf{t}}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}\}$ that wrt to u. How are s and u related? It's sufficient to consider the case when $u = -s$ (the 2 curves have opposite orientations).

How does τ look like for curves that are not necessarily unit speed? In Problem sheet 2, you'll show that:

Theorem 2.5. Let $\boldsymbol{\gamma}(t)$ be a regular 3D curve on which $\kappa \neq 0$. The torsion is given by

$$\tau = \frac{\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'' \cdot \boldsymbol{\gamma}'''}{|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''|^2} \tag{2.9}$$

Every reparametrization of a curve can be made unit-speed, and since these all have the same κ and τ, we deduce that:

Corollary. The curvature κ and torsion τ are both unchanged by reparametrizations (whether unit-speed or not).

This means that κ and τ are fundamental attributes of a curve, independent of its parametrization.

We now show that τ quantifies how much a curve escapes from a 2D plane.

Theorem 2.6. Let $\boldsymbol{\gamma}$ be a regular 3D curve on which $\kappa \neq 0$. The curve is contained in a plane *iff* $\tau = 0$ on the entire curve.

Proof. Without loss of generality, let us work with a unit-speed parametrization $\boldsymbol{\gamma}(s)$ of the curve.

Theorem 2.7. [*Frenet-Serret equations*] Let $\boldsymbol{\gamma}$ be a 3D unit-speed curve on which $\kappa \neq 0$. We have the relations

$$\begin{bmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \\ \dot{\mathbf{b}} \end{bmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{2.10}$$

Proof. We already have the first and third lines from the definitions. To prove the second line, differentiate the cyclic relation, $\mathbf{n} = \mathbf{b} \times \mathbf{t}$.

Example 11. For the helix $\boldsymbol{\gamma}(t) = (\cos t, \sin t, t)$ $(t > 0)$, find the Frenet basis $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, as well as its curvature and torsion.

Example 12. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve. Express $\dddot{\boldsymbol{\gamma}}$ in the Frenet basis. Hence show that

$$\dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}} \times \dddot{\boldsymbol{\gamma}} = \kappa^2 \tau.$$

Example 13. Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve with constant curvature $\kappa > 0$ and $\tau = 0$.

(a) Show that $\boldsymbol{\gamma} + \kappa^{-1}\mathbf{n}$ is a constant vector.

(b) Hence prove that $\boldsymbol{\gamma}$ is part of a circle. Find the radius of the circle.

This is why the quantity $1/\kappa$ is called the radius of curvature (regardless of whether $\boldsymbol{\gamma}$ is a circle or not). It locally quantifies the amount that the curve curves, by comparing it with a 'best-approximating' circle (in 2D) or sphere (in 3D). These are called the *osculating* circle or sphere of $\boldsymbol{\gamma}$.

A couple of concluding remarks to close this Chapter.

- The fact that an anti-symmetric matrix appears in the Frenet-Serret equations is not a coincidence. The reason is found in the fields of *Lie group* and *Lie algebra*...
- In 2D, the signed curvature κ_s specifies curves up to translation and rotation. We have a similar situation in 3D:

  Theorem 2.8. [Fundamental Theorem of Space Curves] Let $\kappa : I \to \mathbb{R}^+$ and $\tau : I \to \mathbb{R}$ be smooth functions ($I \subseteq \mathbb{R}$).

  [Existence] There exists a unit-speed curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ with curvature κ and torsion τ.

  [Uniqueness] If $\tilde{\boldsymbol{\gamma}}$ is another unit-speed curve with curvature κ and torsion τ, then $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are related by a sequence of rotation and translations in $\mathbb{R}^3$.

- Calculating κ and τ for a given curve $\boldsymbol{\gamma}$ can be a messy business without a plan. Here's a helpful flowchart.

Chapter 3

Global properties of plane curves

We now focus on *global* properties of 2D curves: meaning that we're no longer concerned with local properties like the value of κ or τ at a certain point, but rather their *macroscopic* properties, *i.e.* properties of the curve as a whole. For example, we will prove that the area enclosed by a closed curve of a given length is maximised *iff* it's a circle.

We will only deal with curves in $\mathbb{R}^2$ in this Chapter.

We will study 3 theorems in this Chapter: 1) Hopf's *Umlaufsatz*, 2) The Isoperimetric Inequality, and 3) The Four-Vertex Theorem.

Definition. A 2D curve $\boldsymbol{\gamma}$ is said to be _______________ if there exists a constant $L > 0$ such that, for all $t \in \mathbb{R}$,

$$\boldsymbol{\gamma}(t + L) = \boldsymbol{\gamma}(t) \tag{3.1}$$

The *smallest* such L is called the _____________ of the curve.

Definition. A curve is said to be __________ if it has a periodic regular parametrization.

Example 1. Show that the unit circle is closed.

Not every parametrization of closed curves is periodic. For example,

$$\boldsymbol{\gamma}(t) = (\cos t^2, \sin t^2),\ t \in \mathbb{R},$$

is a parametrization of the unit circle, but we can't find the period L in this case (as t increases, the circle closes up increasingly fast - so we can't find the smallest L).

But if a closed curve is parametrized by its *arc-length*, then we can show that it *is* periodic, and the period is equal to its ______________ (unsurprisingly).

Lemma 3.1. If $\boldsymbol{\gamma}$ is a closed curve, then its arc-length parametrization is periodic.

Proof. By definition $\boldsymbol{\gamma}$ has a periodic parametrization: let's call it $\boldsymbol{\gamma}(t)$ with period T,*i.e.*

$$=$$

Now consider its arc-length parametrization, $\tilde{\boldsymbol{\gamma}}(s)$ (let's put a tilde whenever the curve is considered to be a function of s). The goal is to show that

$$=$$

where L is the length of the entire closed curve:

$$L =$$

First note that s and t are related by the arc-length function.

$$s(t) = \qquad\qquad \implies s(t+T) =$$

$$\implies s(t+T) - s(t) =$$

Letting $v = u - t$, the integral becomes

$$s(t+T) - s(t) =$$

So we have: $s(t+T) = s(t) + L$.

To conclude, observe the following changes of parameters:

The previous Lemma means that from now on, when dealing with a closed curve, we can just work with its arc-length parametrization, which, we know, has period L and is also regular by Theorem 1.7.

Definition. A closed curve is __________ if it has no self-intersection.

Question. Sketch a closed curve which is simple and one which is not simple.

Equivalently, we could say that if a closed curve $\boldsymbol{\gamma}$ has period L, then it is *simple* if $\boldsymbol{\gamma}$ is ________________.

3.1 Hopf's *Umlaufsatz*

This section concerns the 'total' signed curvature as you go round a closed curve, *i.e.*

(3.2)

(we can just work with the arc-length parametrization). Why is this interesting? Well, note that for unit-speed curves, we can write the signed curvature as the derivative of the turning angle (Lemma 2.3), so that

(3.3)

where L is the length of the closed curve. So this integral actually measures the total turning angle which the tangent sweeps out as the curves comes back onto itself.

In the case of the circle, you can guess the answer.

Example 2. Calculate $\oint \kappa_s \, \mathrm{d}s$ for $\boldsymbol{\gamma}(t) = (R\cos t, R\sin t)$.

Can you think of a closed curve which winds around *more than once* before closing up? In this case, you can imagine that $\oint \kappa_s \, \mathrm{d}s > 2\pi$.

Lemma 3.2. Let $\boldsymbol{\gamma}$ be a unit-speed closed curve, then the total signed curvature is a multiple of _____, *i.e.*

$$\oint \kappa_s \, \mathrm{d}s = \qquad\qquad\qquad \text{for some } N \in \mathbb{Z}$$

Proof. Again let's work with the turning angle and arc-length parametrized $\boldsymbol{\gamma}$. Let L be the period. As before, this means:

$$\oint \kappa_s \, \mathrm{d}s = \tag{3.4}$$

and we want to show that this is a multiple of 2π.

You can see that the above conclusion would follow from the fact that the tangent vector $\dot{\boldsymbol{\gamma}}$ of a periodic curve will also come back onto the same vector as the curve closes up. Let's prove this properly.

Since $\boldsymbol{\gamma}(s)$ is unit speed, it has period equals to the arc-length L (by Lemma 3.1). This means that $\boldsymbol{\gamma}(s+L) =$ __________ for all s. We now show that the unit tangent $\mathbf{t}$ is also L-periodic, and, in particular, $\mathbf{t}(s=0) = \mathbf{t}(s=L)$.

In terms of the turning angle, $\mathbf{t}(s) = (\cos\varphi(s), \sin\varphi(s))$, so

This implies that $\varphi(L) - \varphi(0)$ is a multiple of 2π. □

Definition. The integer N in Lemma 3.2 is called the ____________________ of $\boldsymbol{\gamma}$.

Intuitively, N is the number of times the curve goes around itself before closing up.

Example 3. Draw closed curves with rotation index $N = 0, \pm 1, \pm 2, 3$.

You might have noticed that other than the cases $N = \pm 1$, the curves have self-intersections, *i.e.* they are not simple. This important observation is known as...

Theorem 3.3. [*Hopf's Umlaufsatz*] If $\boldsymbol{\gamma}$ is a simple closed curve in $\mathbb{R}^2$, then its rotation index is either ______________, *i.e.*

$$\oint \kappa_s \, \mathrm{d}s = \pm 2\pi.$$

In German, *Umlauf* means rotation or circulation, and *Satz* means theorem. It is named after the German mathematician Heinz Hopf (1894-1971).

The proof is quite long and technical and we won't go into it in this course[1], however, you are expected to be able to state what the theorem says. You should have an intuitive feel of what it means.

Question. How can you tell whether a curve has rotation index 1 or -1?

Definition. A simple closed curve is ________________________ if it has rotation index _____. Otherwise it is *negatively oriented*.

Example 4. Let $\boldsymbol{\gamma}(t)$ be a circle radius 2 centred at the origin. Find $\oint \kappa_s(t) \, \mathrm{d}t$. Comment on the answer in light of the Umlaufsatz.

[1]See Bär, page 41.

Example 5. By applying Hopf's Umlaufsatz to the curve $\boldsymbol{\gamma}(t) = (p\cos t, q\sin t)$, (where $p > 0$ and $q > 0$) evaluate

$$\int_0^{2\pi} \frac{\mathrm{d}t}{p^2\sin^2 t + q^2\cos^2 t}.$$

3.2 Isoperimetric Inequality

What is the maximum area that can be formed with a given length of rope?

This is the famous *isoperimetric problem*. Our intuition suggests that the solution is a circle, but proving it is surprisingly difficult. The earliest attempt was probably by Zenodoros around 200BC[2] but a rigorous proof was not established until the 19th century. Here we will study an elegant proof based on differential geometry.

Phrasing the problem in terms of calculus, we are interested in the area inside a *simple* closed curve $\boldsymbol{\gamma}$. We denote this area by the double integral

$$A = \tag{3.5}$$

where Ω is the finite area bounded by $\boldsymbol{\gamma}$.
Here are a few results we will need for the proof. Remember this from Vector Calculus?

Theorem 3.4. [*Green's Theorem*] Let Ω be the region in the plane bounded by a simple closed curve $\boldsymbol{\gamma}$, which is positively oriented. Let $P(x, y)$ and $Q(x, y)$ be functions with continuous 1st-order partial derivatives in Ω, then

$$\iint_\Omega (Q_x - P_y)\, \mathrm{d}x\, \mathrm{d}y =$$

Example 6. Show that the area bounded by the unit-speed curve $\boldsymbol{\gamma}(s) = (x(s), y(s))$ can be written as

$$A = \int_0^L x\dot{y}\, \mathrm{d}s,$$

where L is the length of $\boldsymbol{\gamma}$. Write down two other expressions for the area.

[2] For an interesting historical account of the isoperimetric and other optimization problems, I recommend *The Parsimonious Universe* by Hildebrant and Tromba, as well as this video lecture by John Barrow: `http://bit.ly/1Jwrz7g`

Lemma 3.5. For all $A, B, C, D \in \mathbb{R}$,

$$(AB - CD)^2 \leq (A^2 + C^2)(B^2 + D^2),$$

where the equality holds iff $AD + BC = 0$.

Lemma 3.6. ["GM $\leq$ AM"] For all $A, B \geq 0$, we have

$$\sqrt{AB} \leq \frac{A+B}{2}.$$

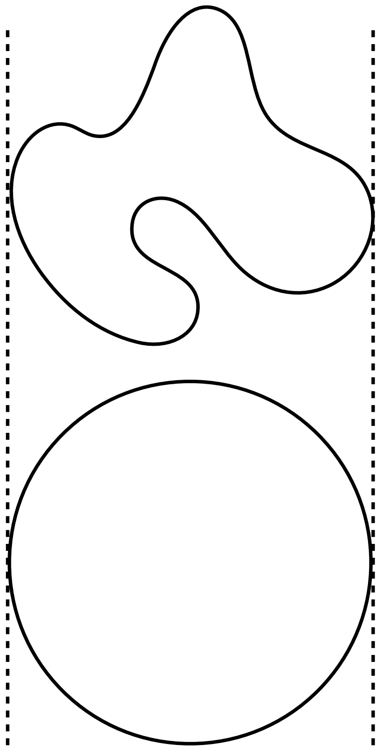
When does equality hold?

Theorem 3.7. [Isoperimetric Inequality] Let $\boldsymbol{\gamma}$ be a simple closed curve with length L and enclosed area A. Then

$$A \leq \frac{L^2}{4\pi},$$

and equality holds iff $\boldsymbol{\gamma}$ is a circle.

Proof. This proof was published in 1939 by Erhard Schmidt.



Parametrization of the given curve by arc-length as:

$$\boldsymbol{\gamma}(s) =$$

Place the origin at the centre of a circle C (which does not intersect $\boldsymbol{\gamma}$) with diameter equal to the distance between the two vertical lines which meet and bound $\boldsymbol{\gamma}$ at $s = 0$ and $s = s_1$. Also assume that C and $\boldsymbol{\gamma}$ are both positively oriented.
Let the equation of the circle be

$$C(s) = \qquad\qquad .$$

Note that there is no guarantee that the circle is unit-speed in this parametrization.
Also, we are free to choose ____________________.

Using Green's Theorem (see Example 6), the area enclosed by $\boldsymbol{\gamma}$ (length L) is

$$A =$$

Similarly, the area enclosed by the circle (say, radius r),

$$\pi r^2 =$$

Adding the two areas gives:

$$A + \pi r^2 =$$

Now apply the "GM$\leq$ AM" inequality to the areas of $\boldsymbol{\gamma}$ and the circle.

This proves the Isoperimetric Inequality. It remains to show that equality holds iff $\boldsymbol{\gamma}$ is a circle. The "if" part is immediate. Let's suppose $A = L^2/4\pi$. We want to show that $(x(s), y(s))$ describes a circle.

Firstly, note that the "GM$\leq$AM" becomes an equality *iff*:

The other inequality (from Example 3.5) becomes an equality *iff*

But we also know that $x^2 + \bar{y}^2 = r^2$. Differentiating wrt s gives:

In summary, the Isoperimetric *Equality* holds *iff* $\boldsymbol{\gamma}(s) = (x(s), \bar{y}(s)+ \text{ constant})$. Since $C = (x(s), \bar{y}(s))$ describes a circle, $\boldsymbol{\gamma}$ is also a circle (translated along the y direction). $\square$

Note that the proof can be generalised so that $\boldsymbol{\gamma}$ only needs to be *piecewise* continuous.

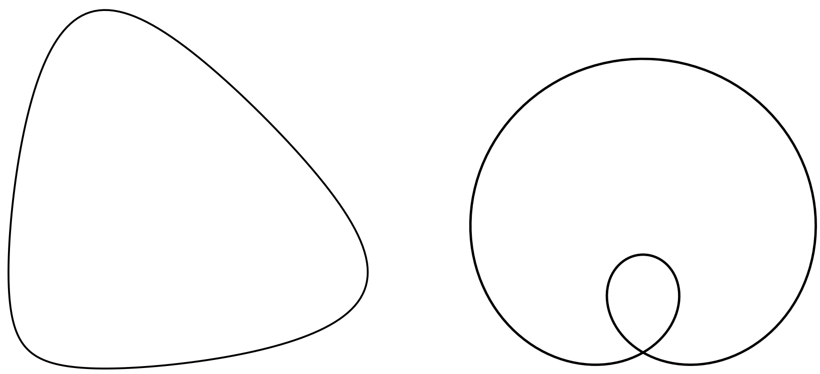
3.3 The Four-Vertex Theorem

Definition. A *vertex* of a curve $\boldsymbol{\gamma}(t) \subset \mathbb{R}^2$ is defined to be a point where the signed curvature attains a stationary point, *i.e.*

$$=$$

So a vertex is where the curve bends 'most' or 'least' *locally.* A circle has constant curvature, so every point on the circle can be regarded as a vertex. But think about an ellipse. How many vertices does it have?

Example 7. How many vertices do these figures have? Plot $\kappa_s(s)$ in each case.



The last Example shows a closed curve with 2 vertices. The curve is clearly not simple. Can you think of a *simple* closed curve with less than 4 vertices? The *Four-Vertex Theorem*, which we will prove, says that there is no such curve.

The Theorem actually concerns any simple closed curve, but in this course we will only prove a less general, though still useful, result for a subset of simple closed curves.

Definition. A closed curve $\boldsymbol{\gamma}$ is said to be _______________ if the line segment joining any two points on $\boldsymbol{\gamma}$ lies *entirely* in the interior[3] of $\boldsymbol{\gamma}$.

[3] It might seem *obvious* that that a simple closed curve divides the plane into two regions: one bounded (the interior), and the other unbounded (the exterior), but the proof is rather involved. This is the *Jordan Curve Theorem*, which you will meet in a topology course.

Example 8. Draw a convex and a non-convex curve.

Lemma 3.8. Let $\boldsymbol{\gamma}$ be a convex simple closed curve which is not a circle. If $\boldsymbol{\gamma}$ has up to 3 vertices, then there is a straight line dividing $\boldsymbol{\gamma}$ into two open segments: one on which $\dot{\kappa}_s > 0$ and on the other $\dot{\kappa}_s \leq 0$ (or $\dot{\kappa}_s \geq 0$ on one and $\dot{\kappa}_s < 0$ on the other).

Proof. : Parametrize $\boldsymbol{\gamma}(s)$ by arc-length and let L be its length. As $\boldsymbol{\gamma}$ is assumed to be smooth, $\kappa_s : [0, L] \to \mathbb{R}$ is a continuous function, and so it must attain a maximum and a minimum somewhere on $[0, L]$ (___________________ theorem). This shows that $\boldsymbol{\gamma}$ has at least _____ vertices. Call these points $\mathbf{p}$ and $\mathbf{q}$ respectively.

If $\mathbf{p} = \mathbf{q}$ (max=min) then κ_s is constant on the whole curve, *i.e.* it is a __________.

Assume $\mathbf{p} \neq \mathbf{q}$. Suppose $\mathbf{p}$ and $\mathbf{q}$ are the *only* vertices. Draw a straight line through $\mathbf{p}$ and $\mathbf{q}$. This line will separate $\boldsymbol{\gamma}$ into *two* segments (since $\boldsymbol{\gamma}$ is __________). Since $\mathbf{p}$ and $\mathbf{q}$ are the maximum and minimum of κ_s, then κ_s must be increasing on one segment (*i.e.*____________), and decreasing on the other (____________), so the Lemma holds.

It remains to consider the case when $\boldsymbol{\gamma}$ has 3 vertices, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$. By the same logic, κ_s on the three segments must be either increasing or decreasing. Since there must be two adjacent increasing (or decreasing) segments (with $\dot{\kappa}_s = 0$ at the joints), the Lemma still holds. □

This rather strange Lemma that we just proved simply says that any curve with up to 3 vertices must comprise two pieces on which $\dot{\kappa}_s$ has opposite signs. We will use it to prove. . .

Theorem 3.9. [*Four-Vertex Theorem*] Every convex simple closed curve in $\mathbb{R}^2$ which is not a circle has at least 4 vertices.

Proof. Suppose that $\boldsymbol{\gamma}(s)$ (parametrized by arc-length) has up to 3 vertices. We will show that a contradiction follows, and therefore $\boldsymbol{\gamma}$ must have at least 4 vertices.

- By Lemma 3.8, there is a line ℓ which divides $\boldsymbol{\gamma}$ into two segments on which $\dot{\kappa}_s$ has opposite signs (with $\dot{\kappa}_s = 0$ only at the vertices).

- Let's set up the coordinate axes. Translate the curve along so that the line ℓ coincides with the x axis, with the origin somewhere within the curve (Remember: translation and rotation don't change κ_s). Furthermore, we can place the segment with $\dot{\kappa}_s \geq 0$ on top (above the x axis). Let's draw the figure below.

- Let's focus on the y-coordinate of $\boldsymbol{\gamma}(s) = (x(s), y(s))$.

  It's worth recalling that $\ddot{\boldsymbol{\gamma}} = \kappa_s \mathbf{n}_s$. Writing this in terms of $(x(s), y(s))$:

- Now from the setup shown in the figure, we see that the combination $\dot{\kappa}_s y$ is always non-negative. In fact, on the segment below the x axis, _______________.

  This implies that the integral

$$I = \int_0^L \dot{\kappa}_s \, y \; \mathrm{d}s \tag{3.6}$$

- However, integrating I by parts, and using the fact that x, y, κ (and their derivatives) are L-periodic, we find

  This contradicts Eq. 3.6, which says that $I > 0$. $\square$

Example 9. Consider the curve $\boldsymbol{\gamma}(t) = (p\cos t, q\sin t)$. From Example 5, its signed curvature is given by

$$\kappa_s(t) = \frac{pq}{[(p\sin t)^2 + (q\cos t)^2]^{3/2}}.$$

How many vertices does $\boldsymbol{\gamma}$ have?

Chapter 4

Surfaces

Curves and surfaces live in 3D. But what distinguishes one from the other?

Think of a curve (whether 2D or 3D) as something that looks almost like a line *locally*. A line is one-dimensional, so it's possible to describe a curve with one parameter.

Similarly, a surface is something that's looks almost like a __________ locally, and so requires _____ parameters to describe it. We'll shortly define it formally.

The formal definition of surfaces is quite technical. Why? Well, as an analogy, we can easily understand what a *function* does, but it takes a bit of work to define them formally. With the formal definition, we were able to understand why, for instance, the circle is not a function, but is actually two functions that can be patched together.

Similarly, even though we think we know what looks like a surface, it takes a bit of work to define them formally, but when we get there we will understand why the sphere is (or is not?) a surface.

4.1 Prelude

Here's a bunch of things we need. Make sure you are familiar with them.

Definition. For $\mathbf{x} \in \mathbb{R}^n$, the *Euclidean norm* (or L^2 norm) in $\mathbb{R}^n$ is defined as

$$|\mathbf{x}| = \tag{4.1}$$

Make sure you use the right norm in the right dimension.

Definition. An *open ball*, $B_\varepsilon(\mathbf{x_0})$ in $\mathbb{R}^n$, is defined as the set

$$B_\varepsilon(\mathbf{x_0}) = \tag{4.2}$$

where $\varepsilon > 0$ and $\mathbf{x_0} \in \mathbb{R}^n$.

The norm in this definition is the n-dimensional Euclidean norm. In Analysis, you may have called $B_\varepsilon(\mathbf{x_0})$ the ______________________ of $\mathbf{x_0}$.

Definition. The set $S \subseteq \mathbb{R}^n$ is said to be *open* if $\forall \mathbf{x} \in S$, $\exists \varepsilon > 0$ such that ___________.

Definition. A set is said to be __________ if it is not open.

Definition. A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\} \subset \mathbb{R}^n$ is said to be *linearly independent* if the equation

only has the trivial solution $c_1 = c_2 = \ldots = c_n = 0$.

Although linear independence is strictly a property of the *set* $\{\mathbf{v}_i\}$, a widespread abuse of terminology (but a rather convenient one) allows us to say that the *vectors* $\mathbf{v}_i$ themselves are linearly independent.

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *continuous* at $\mathbf{x_0}$ if, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\Longrightarrow \tag{4.3}$$

Note that the norms in the line above may be different.

Definition. A function $f : A \to B$ is said to be __________, or C^∞, at $\mathbf{x_0}$ if f has continuous partial derivatives of all orders on A.

Definition. The *grad* operator acting on $f(x, y, z)$ is defined as $\nabla f =$ _______________.

Theorem. The vector __________ is normal to the surface $f(x, y, z) =$ constant. (For proof, see Chapter 6 (§2.1) of Vector Calculus.)

We will be looking at a map from one vector space to another. The thing that tells you whether such a map is well-behaved (has derivatives, invertible) is the _______________.

Definition. The *Jacobian matrix* for the map $f : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix

$$J = \frac{\partial f_i}{\partial x_j} = \tag{4.4}$$

When $n = m$, J is a square matrix. Its determinant, $|J|$ (sometimes also confusingly called "the Jacobian") determines whether the inverse function exists *locally*.

Theorem. [*Inverse Function Theorem*] Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function. If the Jacobian $|J| \neq 0$ at a point $\mathbf{x_0} \in \mathbb{R}^n$, then there exists a neighbourhood U of $\mathbf{x_0}$ such that f^{-1} exists and is smooth.

(compare the above with Lemma 1.6). In multi-variable calculus, the Jacobian plays an analogous role to the derivative in one-variable calculus.

Here are some objects which are prototype of surfaces (yet to be defined).

Example 1. Let $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ be unit vectors in $\mathbb{R}^3$ that are linearly independent. Let Π be the plane which is parallel to these vectors, and passes through the point $\mathbf{a}$. Then Π can be written as the function $\boldsymbol{\sigma} : \mathbb{R}^2 \to \Pi$, with

$$\boldsymbol{\sigma}(u, v) = \tag{4.5}$$

where $(u, v) \in \mathbb{R}^2$.

Try not to think of the plane as a geometric object, but as the image of a *mapping* from $\mathbb{R}^2$ to $\mathbb{R}^3$.

If we know an image point (in $\mathbb{R}^3$) of this mapping, is it possible to work out the *preimage* in the domain $\mathbb{R}^2$? If we can, then this would be the inverse function $\boldsymbol{\sigma}^{-1} : \Pi \to \mathbb{R}^2$.

Example 2. Assuming that $\hat{\mathbf{p}} \cdot \hat{\mathbf{q}} = 0$, find an explicit expression for $\boldsymbol{\sigma}^{-1}$ (the inverse plane mapping).

Example 3. Sketch the image of the function $z = \sqrt{x^2 + y^2}$. Express it in the form $\boldsymbol{\sigma} : \mathbb{R}^2 \to \mathbb{R}^3$.
Describe $\boldsymbol{\sigma}^{-1}$ in one word.

Example 4. Using spherical coordinates, write down the equation of the unit sphere in the form $\boldsymbol{\sigma} : U \to \mathbb{R}^3$, where $U \subseteq \mathbb{R}^2$. Does $\boldsymbol{\sigma}^{-1}$ exist for your choice of U?

Example 5. Using cylindrical coordinates, write down the equation of a unit cylinder in the form $\boldsymbol{\sigma} : U \to \mathbb{R}^3$, where $U \subseteq \mathbb{R}^2$. Is $\boldsymbol{\sigma}^{-1}$ continuous for your choice of U?

As mentioned in the introduction, a surface can be defined as a 3D object which is *locally* 2-dimensional. The previous Examples inspire us to define a surface via a map from some subset of $\mathbb{R}^2$ to a subset of $\mathbb{R}^3$. We want this map to have 'nice' properties (*e.g.* continuous, invertible).

Definition. Two sets A and B are said to be ____________________ if there exists a function $f : A \to B$ such that

- f is _______________ on A
- f is _______________
- $f^{-1} : B \to A$ is also _______________.

Such a function f is called a ____________________.

Intuitively, homeomorphic objects have 'essentially' the same shape. Think of a homeomorphism can as a sequence of continuous stretching and bending, but not tearing or gluing (*e.g.* a book is homeomorphic to a ball, a donut is homeomorphic to a tea cup, but a donut is not homeomorphic to a book.).

Example 6. Prove that the interval $(-1, 1)$ is homeomorphic to $\mathbb{R}$.

4.2 Definition of a surface

We want to define a surface to be sufficiently *nice* objects that we can do calculus on them. It turns out that the following somewhat technical definition, based on homeomorphisms between open sets, is sufficient.

Definition. A subset $S \subset \mathbb{R}^3$ is called a *surface* if, $\forall \mathbf{p} \in S$, there exists an open set $U \subseteq \mathbb{R}^2$, and an open set $V \subseteq \mathbb{R}^3$ containing $\mathbf{p}$, such that U is homeomorphic to $S \cap V$.

Definition. The homeomorphism $\boldsymbol{\sigma} : U \to S \cap V$ is called a ____________________ of the surface (or a *coordinate patch*, or a *chart*).

Example 7. Is the plane Π (Eq. 4.5) a surface?

Note that a surface may have more than one patch/parametrization (as long as one exists at every point of S). This kind of 'patching' can also be done with curves.

Example 8. Describe the unit circle as a union of the images of homeomorphisms on open sets.

Definition. The collection of all coordinate patches whose images cover the whole of S (the patches may overlap) is called an ____________of S.

Example 9. Construct the atlas for the unit cylinder. (See sheet 4 for an alternative atlas.)

Example 10. Show that the double cone $x^2 + y^2 = z^2$ is not a surface.

Proof. Suppose the double cone is a surface, S. Place the vertex at the origin in $\mathbb{R}^3$ and parametrize the cone using $\boldsymbol{\sigma}$ such that ____________for some $\mathbf{a} \in \mathbb{R}^2$. By definition of a surface, there exists an open set $U \subseteq \mathbb{R}^2$ containing $\mathbf{a}$ which is mapped onto some open set, $S \cap V$ around the vertex.

Clearly $S \cap V$ contains a point $\mathbf{p}$ in the lower cone, and $\mathbf{q}$ in the upper cone. These correspond to two points in U, namely, __________ and __________.

Now draw a smooth curve $\boldsymbol{\gamma}$ joining $\boldsymbol{\sigma}^{-1}(\mathbf{p})$ to $\boldsymbol{\sigma}^{-1}(\mathbf{q})$ *without going through* $\mathbf{a}$. This 2D curve in U will be mapped onto a 3D curve $\boldsymbol{\sigma}(\boldsymbol{\gamma})$ lying on the surface, joining $\mathbf{p}$ to $\mathbf{q}$ continously without passing through the vertex. This contradicts the ______________________________.

4.3 Regular surfaces

Recall why *regular* curves are important: since $\boldsymbol{\gamma}'(t) \neq \mathbf{0}$, a tangent vector is well defined at every point. Also, the entire curve can be swept out as t increases, never stopping and never backtracking (see sheet 1 Q4). We want a similar property for surfaces.

Unlike a curve, which has a unique tangent vector at a given point, there are clearly many possible tangent vectors at a given point on a surface.

Definition. A *tangent vector* to a surface S at $\mathbf{p} \in S$ is the tangent vector of any curve on S passing through $\mathbf{p}$.

Definition. The *tangent space* (or *tangent plane*), __________, is the set of *all* tangent vectors at $\mathbf{p}$.

Definition. A surface parametrized by $\boldsymbol{\sigma}(u, v)$ is said to be *regular* if

- $\boldsymbol{\sigma}$ is smooth (*i.e.* it has partial derivatives of all orders), and
- $\boldsymbol{\sigma}_u \equiv \frac{\partial \boldsymbol{\sigma}}{\partial u}$ and $\boldsymbol{\sigma}_v \equiv \frac{\partial \boldsymbol{\sigma}}{\partial v}$ are __________________________.

Let's try to make sense of this definition. The first condition is needed so we can do calculus on the surface. What about the second?

The vectors $\boldsymbol{\sigma}_u(\mathbf{p})$ and $\boldsymbol{\sigma}_v(\mathbf{p})$ are two tangent vectors at the point $\mathbf{p}$. If the surface is regular, then a tangent plane can be drawn at $\mathbf{p}$, with $\boldsymbol{\sigma}_u(\mathbf{p})$ and $\boldsymbol{\sigma}_v(\mathbf{p})$ lying on the plane. So, the second condition simply ensures that the two tangent vectors span a plane.

In summary, regular curves have well-defined tangent vectors. Regular surfaces have well-defined tangent _____________.

Proposition 4.1. Let $\boldsymbol{\sigma}(u, v)$ be a smooth parametrization of a surface. The surface is regular if and only if $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \neq \mathbf{0}$ on the surface.

Proof. This is a simple exercise in linear algebra (see sheet 4).

Example 11. Show that the plane (Eq. 4.5) is a regular surface.

Example 12. Show that the unit cylinder is a regular surface.

Example 13. Sketch some surfaces that are not regular.

Example 14. Show that the surface parametrized by

$$\boldsymbol{\sigma}(u,v) = \left(\sqrt{1-v}\,\cos u, \sqrt{1-v}\,\sin u,\, v\right)$$

is regular. Find the equation of its tangent plane at $\mathbf{p}$ where $(u,v) = (\pi/4, 0)$.

If the surface has a closed-form expression $f(x, y, z) = 0$, then the next Theorem gives us an even quicker way to check if a surface is regular.

Theorem 4.2. Suppose a surface, S, is defined by the equation

$$f(x, y, z) = 0.$$

where f is smooth. If ______________ then S is a regular surface.

Let's think about why this theorem makes sense.

Suppose we have a surface $\boldsymbol{\sigma}(u, v)$, which can also be expressed as $f(x, y, z) = 0$. We have two ways to express the normal to the surface at a given point: ______________ and _______. A surface is regular if neither of these expression is $\mathbf{0}$ (by Prop. 4.1).

The proof is slightly technical (involving the Inverse Function Theorem) and I won't go into here (See Pressley, page 118). We shall use this result without proof.

Caution! The converse does not hold! If $\nabla f = \mathbf{0}$, no conclusion about the surface can be drawn (see sheet 4).

Example 15. Redo the previous example using Theorem 4.2. Sketch the surface.

Example 16. Use Theorem 4.2 to determine whether the unit cylinder and the double cone are regular surfaces.

Example 17 (*the graph*)**.** If f is smooth function on U, prove that $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ where

$$\boldsymbol{\sigma}(u, v) = (u, v, f(u, v)),$$

is a regular surface. Describe the parametrization in words.

4.4 Surface reparametrization

Just like curves, surfaces can be parametrized (patched) in many ways. But do we change the properties of a surface by a reparametrization?

Recall from Chapter 1 that when a regular curve is reparametrized, it is again regular. We will prove a similar result for surfaces here. But first, some revision on the Jacobian.

Example 18. Describe the surface parametrized by

$$\boldsymbol{\sigma}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)\,, \quad (u,v) \in (0,2\pi) \times (0,\pi/2).$$

(a) Is $\boldsymbol{\sigma}$ a regular parametrization of the surface?

(b) Write down a Cartesian parametrization $\tilde{\boldsymbol{\sigma}}(\tilde{u},\tilde{v})$ of the same surface.

(c) Write down a smooth map Φ such that $\Phi(u,v) = (\tilde{u},\tilde{v})$.

(d) Find the Jacobian determinant, $|J|$, for the transformation Φ. Is Φ invertible?

The definitions below are very similar to those we have studied for curves.

Definition. $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ is said to be a ____________________ of a surface $\boldsymbol{\sigma}(u, v)$ if there is a bijective map Φ such that

$$ \tag{4.6}$$

where Φ and Φ^{-1} are both smooth.

Definition. Two sets A and B are said to be ____________________ if there exists a function $\Phi : A \to B$ such that

- Φ is smooth
- Φ is invertible
- Φ^{-1} is also smooth

Such a function Φ is called a ____________________.
(compare this with the definition for homeomorphism. In other words, a diffeomorphism is a smooth homeomorphism.)

Definition. A diffeomorphism Φ linking two parametrizations of a surface is known as a *reparametrization* map or a _______________ map.

Proposition 4.3. A reparametrization of a regular surface is again regular

Proof. Let $\boldsymbol{\sigma}(u, v)$ be a regular parametrization of a surface. Let the reparametrization map be $\Phi(\tilde{u}, \tilde{v}) = (u, v)$, where $u = u(\tilde{u}, \tilde{v})$ and $v = v(\tilde{u}, \tilde{v})$.

To check that $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ is a regular surface, we need to check two conditions, namely:

Since both Φ and $\boldsymbol{\sigma}$ are smooth, their composition is smooth.

To find $\tilde{\boldsymbol{\sigma}}_{\tilde{u}}$ and $\tilde{\boldsymbol{\sigma}}_{\tilde{v}}$, we apply the Chain Rule:

Since $\boldsymbol{\sigma}$ is regular, we know that _______________. Also, since Φ is a diffeomorphism, _______________ (this is the converse of the Inverse Function Theorem).

Hence $\tilde{\boldsymbol{\sigma}}_{\tilde{u}} \times \tilde{\boldsymbol{\sigma}}_{\tilde{v}} \neq \mathbf{0}$, and so $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ is again a regular surface. □

Let's compare this with what happened in Chapter 1 for curves. There was no need to talk about a regular *parametrization* of a curve: once a curve is regular, its reparametrizations are again regular. If we find a dodgy parametrization of a regular curve which produces a non-regular curve, then that parametrization cannot be a 'reparametrization' (*i.e.* not a diffeomorphism).

As we've just shown, the analogous results for surfaces are as follows:

Chapter 5

Surfaces II – Examples

5.1 Quadrics

The quadric surfaces are a 3D generalisation of the quadratic equation $ax^2 + bx + c = 0$.

Question. Guess the form of the quadric $f(x, y, z) = 0$.

Definition. A quadric is an equation of the form

$$\mathbf{x}^T A\mathbf{x} + \mathbf{b} \cdot \mathbf{x} + c = 0, \tag{5.1}$$

where $\mathbf{x} = (x \;\; y \;\; z)^T$, A is a symmetric 3×3 matrix, $\mathbf{b}$ is a constant vector, and $c \in \mathbb{R}$. One can easily multiply out the products to find that Eq. 5.1 is equivalent to the equation we guessed.

Example 1. Put the following in quadric form (5.1): a) the unit sphere, b) the double cone, c) the origin.

A general quadric with lots of cross terms can be reduced to a simpler form using a sequence of rotations and translations[1]. In fact, any quadric can be reduced one of the 14 *standard quadrics* listed on the next page.

You should find your own way of remembering their names. Clues to their shapes can be found by setting $x = 0$ or $y = 0$ or $z = 0$. We will study many interesting properties of quadrics in the upcoming chapters.

Here are some real-life quadrics. To which class do they belong?

[1]See Pressley page 101 for the proof based on the ideas outlined here.

Quadric	Equation $(a, b, c \neq 0)$
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Double elliptic cone	
	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
Elliptic cylinder	
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
	$y = \frac{x^2}{a^2}$
	$x = 0$
	$x^2 = a^2$
	$x^2/a^2 - y^2/b^2 = 0$
	$x = y = 0$
	$x = y = z = 0$

Table 5.1: The 14 standard quadrics.

5.2 Surfaces of revolution

A surface of revolution is generated by taking a curve $\boldsymbol{\gamma}(u)$ in $\mathbb{R}^3$ which is confined to the x-z plane, say

$$=$$

with $f(u) > 0$, and rotating it about the z axis.

We see that every horizontal slice of the surface is a circle of radius __________. This tells us about the x, y components of the 3D surface, which we can parametrize as

$$\tag{5.2}$$

Example 2. Prove a surface of revolution is a regular surface *iff* it is generated by a regular curve.

Example 3. Give equations of two quadrics which are surfaces of revolution.
In each case, write down the equation of the generating curve $\boldsymbol{\gamma}(u)$ in the x-z plane, and the surface parametrization $\boldsymbol{\sigma}(u, v)$.

Definition. The *meridians* of a surface of revolution (5.2) are curves of constant _____. The *parallels* are the circles of constant _____.

Example 4. Prove that parallels and meridians are orthogonal.

(more about the parallels and meridians when we study cartography in the next Chapter.)

5.3 Ruled surfaces

Take a vertical line segment passing through $x = 1$ and rotate it around the z axis, the surface of revolution obtained is part of a ______________.

Take a line segment inclined at 45° in the x-z plane and rotate it around the z axis, the surface obtained is part of a ______________.

In fact, the segment could do all sorts of crazy acrobatics - but the surface it sweeps out is still essentially a *union of straight lines.*

This kind of surface is called a *ruled surface.*

Definition. A *ruled surface* is parametrized by the equation

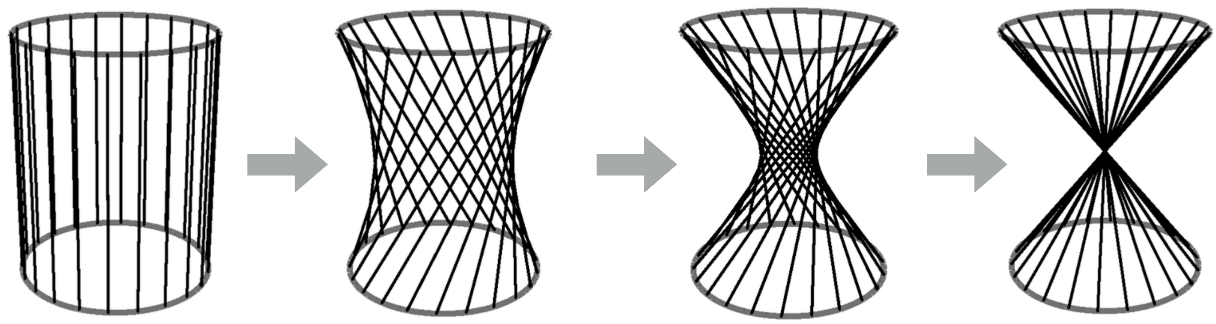
$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\mathbf{a}(u), \tag{5.3}$$

where $\boldsymbol{\gamma}$ and $\mathbf{a}$ are curves in $\mathbb{R}^3$.

Definition. The straight lines that generate the ruled surface are called _______________ (these are lines of constant _____). The curve $\boldsymbol{\gamma}$ (with $v = 0$) is called the ____________________ (or *directrix*).

Example 5. a) Show that the elliptic cylinder and the cone are ruled surface.
b) Show that $\boldsymbol{\sigma}(u, v) = (\cos u - v \sin u,\ \sin u + v \cos u,\ v)$ represents a ruled surface. Which quadric is this?

The picture below shows how the cylinder, hyperboloid and cone can all be generated by twisting a pair of parallel rings joined by taut strings[2].



In fact, twisting the rings the other way would give the same surfaces, but with the rulings going in the opposite direction. This means that the hyperboloid of 1 sheet can be generated by two different families of straight lines, and is said to be ____________________. More about this in problem sheet 5.

(You can see this fact on the picture of the Corporation Street Bridge.)

[2]see a nice animation of this at `http://bit.ly/1iHcnio`

5.3.1 The Möbius strip

Let's finish this Chapter with a very special ruled surface, the Möbius strip[3]: Take a vertical line segment in the x-z plane, say, the line joining $(1, 0, -1/2)$ to $(1, 0, 1/2)$, and rotate it around the z axis. But let's give the segment a $180°$ rotation before closing up.

It can be shown that this construction leads to the parametrization

$$\boldsymbol{\sigma}(u, v) = \left(\left(1 - u \sin\frac{v}{2}\right)\cos v,\ \left(1 - u\sin\frac{v}{2}\right)\sin v,\ u\cos\frac{v}{2}\right), \tag{$*$}$$

where $u \in (-1/2, 1/2)$ and $v \in (0, 2\pi)$.

Question. Identify the base curve and the rulings of the Möbius strip.

In the problem sheet, you will show that the normal taken as $(u, v) \to (0, 0)$ is different from that when $(u, v) \to (0, 2\pi)$, even though these are normals at the same limit point.

This means that when a normal vector, which initially points 'outward', travels around the strip, it ends up on the same point but seems to be pointing 'inward'. Therefore, the Möbius strip has no well-defined notion of *inside* or *outside*.

This kind of surface on which the normal vector field has discontinuous jumps is called a ______________________________ surface.

[3] August Ferdinand Möbius (1790-1868), German mathematician. His other important contributions include *Möbius transformations* in complex analysis, and *Möbius function* in number theory.

Chapter 6

The First Fundamental Form

We will now study lengths, angles and areas on surfaces, all of which are *intrinsic* geometrical properties of surfaces, *i.e.* measurable by the surface inhabitants who do now know about the global properties of the surface (e.g. whether the surface has a hole).

The *first fundamental form* is the name of the quantity which contains *all* information on the intrinsic geometry of surfaces. Essentially, it is simply the *dot product*, but taken on curved surfaces. The first fundamental form is one of the most important concepts in differential geometry.

6.1 Arc-length on surfaces

Take a curve $\boldsymbol{\gamma}(t)$ lying on a surface parametrized by $\boldsymbol{\sigma}(u, v)$. Let $\mathbf{p}$ be a point on $\boldsymbol{\gamma}$.

The tangent __________ lies on the tangent plane at $\mathbf{p}$ (denoted __________). Recall that this tangent plane is spanned by the basis vectors __________ and __________. Therefore, for some $\lambda, \mu \in \mathbb{R}$, we can express $\boldsymbol{\gamma}'$ as the linear combination

(6.1)

On the other hand, points on the curve can also be expressed in terms of the surface. So, we can treat u, v as functions of t and write

$$\boldsymbol{\gamma}(t) =$$

Differentiating wrt t and applying the Chain Rule, we find

$$\tag{6.2}$$

Comparing with (6.1) gives

$$\lambda = \qquad\qquad , \quad \mu = \qquad\qquad . \tag{6.3}$$

Now, recall from Chapter 1 that the arc-length, s, of $\boldsymbol{\gamma}(t)$ satisfies

$$\frac{\mathrm{d}s}{\mathrm{d}t} =$$

This can be expressed in terms of u, v as

$$\tag{6.4}$$

Lemma 6.1. Let $\boldsymbol{\sigma}(u(t), v(t))$ be the parametrization of a curve $\boldsymbol{\gamma}$ lying on a surface $\boldsymbol{\sigma}$. The arc-length of $\boldsymbol{\gamma}$ between $t = t_0$ and $t = t_1$ is given by

$$\tag{6.5}$$

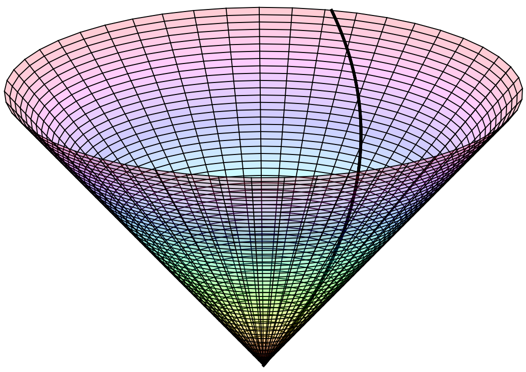
$$\text{where} \quad E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u, \quad F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v, \quad G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v$$

The above result is often expressed in a more traditional form.

Definition. The *first fundamental form* of the surface parametrized by $\boldsymbol{\sigma}(u, v)$ is

$$\mathrm{d}s^2 = \tag{6.6}$$

Example 1. Find the first fundamental form of the cone $\boldsymbol{\sigma}(u, v) = (u \cos v, u \sin v, u)$. Anton the ant lives on the cone. If he walks along the path $u = v = t$ from $t = 0$ to $t = 1$, find an integral expression for the distance he covered.



The first fundamental form helps us measure length on surfaces. This is why it's also known as the ______________ of the surface.

Example 2. Find the first fundamental form of the plane $\boldsymbol{\sigma}(u, v) = \mathbf{a} + u\hat{\mathbf{p}} + v\hat{\mathbf{q}}$, where $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ are orthonormal vectors.
Re-calculate the distance Anton covered if he were living on this plane instead.

Example 3. Find the first fundamental form of the surface of revolution

$$\boldsymbol{\sigma}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

Hence obtain the first fundamental form of the unit sphere parametrized in the usual spherical coordinates.

Example 4. Find the first fundamental form of the unit cylinder.

6.2 The derivative

The fact that the first fundamental forms of the cylinder and the plane are the same is rather curious. What exactly does this mean?

This motivates us to study what happens to points and curves when they are mapped from one surface onto another. This section is slightly technical but very important as it lays the foundation for the rest of the course. Make sure to study this section again.

Consider point $\mathbf{p}$ on a surface S. Recall that the tangent space $T_{\mathbf{p}}S$ is the union of the tangent vectors of *all* curves going through $\mathbf{p}$. Let $\boldsymbol{\gamma}(t)$ be any such curve, and WLOG let $\mathbf{p} = \boldsymbol{\gamma}(t = 0)$.

Take any vector $\mathbf{w}$ in $T_{\mathbf{p}}S$. Clearly $\mathbf{w}$ is a tangent to some curve $\boldsymbol{\gamma}$, so that $\mathbf{w} =$ __________.

We can also express $\mathbf{w}$ in terms of the basis of $T_{\mathbf{p}}S$ (Eq. 6.2) as _________________________ evaluated at $t = 0$.

Let f be a smooth map from S to $\tilde{S}$ (note that f maps 3D vectors to 3D vectors). This means, for instance, that $\mathbf{p} \in S$ is mapped to __________ $\in \tilde{S}$, and a curve $\boldsymbol{\gamma}(t)$ on S is mapped to ______________________________ on $\tilde{S}$. Note that we are using the same parameter t to parametrize both the curve on S and its image on $\tilde{S}$ (see diagram below).

The tangent vector $\tilde{\mathbf{w}}$ to the image curve is given by

$$\tilde{\mathbf{w}} = \qquad (6.7)$$

This vector lies in the tangent space ____________. The map which takes $\mathbf{w}$ to $\tilde{\mathbf{w}}$ (mapping one tangent space to another) is called the _______________, simply denoted as __________ (since the notation in (6.7) is quite cumbersome).

Definition. Given a smooth map $f : S \to \tilde{S}$ and $\mathbf{p} \in S$, the *derivative* $D_{\mathbf{p}}f$ is defined as

$$D_{\mathbf{p}}f : \qquad\qquad\qquad\qquad \text{such that} \quad D_{\mathbf{p}}f(\mathbf{w}) = \tilde{\mathbf{w}}. \tag{6.8}$$

where $\tilde{\mathbf{w}}$ is given by (6.7).

In short, f maps a surface to a surface, but $D_{\mathbf{p}}f$ maps a tangent space to a tangent space. Note that the derivative depends on the map f and the base point $\mathbf{p}$. It's helpful to read $D_{\mathbf{p}}f$ as:

Equation 6.7 looks as though the the derivative also depends on the choice of the curve $\boldsymbol{\gamma}$ through $\mathbf{p}$, but actually this isn't the case.

Lemma 6.2. The derivative $D_{\mathbf{p}}f$ is independent of the choice of the curve $\boldsymbol{\gamma}$ through $\mathbf{p}$.

Proof. The setup is the same as before (refer to previous diagram).

Let $\boldsymbol{\sigma}(u, v)$ be the parametrization of S and let $\mathbf{p} =$ __________ for some curve $\boldsymbol{\gamma}$.

Since $\mathbf{p}$ lies on the surface, we can also write $\mathbf{p} =$ ____________________.

Under the map $f : S \to \tilde{S}$, let $f \circ \boldsymbol{\sigma}(u, v) =$ _____________, where $\tilde{u} = \tilde{u}(u, v)$ and

Now take $\mathbf{w} \in T_{\mathbf{p}}S$. Let's try to express the image, $\tilde{\mathbf{w}}$, in terms of the basis of the new tangent space, namely, __________ and __________.

This is all quite messy, but actually things look much simpler if we recast our findings in terms of *matrices* written in different bases.

Since $\mathbf{w} \in T_{\mathbf{p}}S$, we can express it in terms of the basis of $T_{\mathbf{p}}S$ as

$$\mathbf{w} = \lambda \boldsymbol{\sigma}_u + \mu \boldsymbol{\sigma}_v$$

where $\lambda = \underline{\qquad\qquad}$ and $\mu = \underline{\qquad\qquad}$ (see Eqs. 6.1 and 6.2). If instead, we were to express $\mathbf{w}$ in terms of the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$. It would simply read:

$$\mathbf{w} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}.$$

Similarly, we can express $\tilde{\mathbf{w}}$ in the basis $\{\tilde{\boldsymbol{\sigma}}_{\tilde{u}}, \tilde{\boldsymbol{\sigma}}_{\tilde{v}}\}$. Indeed, our long calculations give:

where J is the usual Jacobian matrix for the transformation $(u, v) \to (\tilde{u}, \tilde{v})$. In this context, J acts as the $\underline{\qquad\qquad\qquad\qquad}$ matrix between bases $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ and $\{\tilde{\boldsymbol{\sigma}}_{\tilde{u}}, \tilde{\boldsymbol{\sigma}}_{\tilde{v}}\}$.

The matrix equation shows that the derivative is independent of the choice of γ. $\square$

Our discovery is so nice it deserves a special mention.

Corollary. The derivative $D_{\mathbf{p}}f$ is a *linear* map which can be represented by the matrix multiplication

$$D_{\mathbf{p}}f(\mathbf{w}) = \tag{6.9}$$

6.3 Induced dot product

Imagine we know all about lengths and angles on one surface S. How are lengths and angles measured on $f(S) = \tilde{S}$? We now show that the derivative gives us the necessary tool to perform these measurements on $\tilde{S}$.

Given our study of the derivative, we can write down how f maps a dot (or *inner*) product on $T_{\mathbf{p}}S$ to a dot product on $T_{f(\mathbf{p})}\tilde{S}$.

$$\langle \mathbf{v}, \mathbf{w} \rangle \equiv \mathbf{v} \cdot \mathbf{w} \rightarrow \tag{6.10}$$

Each side of Eq. (6.10) is a function of any two tangent vectors in $T_{\mathbf{p}}S$. We can express Eq. (6.10) more abstractly in *operator* form (without the vector arguments) as:

$$\tag{6.11}$$

where the subscript f denotes the fact that this dot product is __________ by f. Each of these functions takes a pair of vectors and spits out a number.

We can similarly define the *induced norm* $|\cdot|_f$ as follows. Denote the norm $|\mathbf{w}| \equiv \sqrt{\langle \mathbf{w}, \mathbf{w} \rangle}$. Under the map f, the *induced norm* is defined as:

Lemma 6.3. For any vector $\mathbf{v}$ and $\mathbf{w}$,

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{2} \left(|\mathbf{v} + \mathbf{w}|^2 - |\mathbf{v}|^2 - |\mathbf{w}|^2 \right)$$

6.4 Local isometry

We now show that if f *preserves* the dot product, then this is equivalent to the fact that f preserves the first fundamental form, and hence the length of curves on surfaces.

Definition. Let S and $\tilde{S}$ be surfaces. The map $f : S \to \tilde{S}$ is called a ____________________ if, at every point $\mathbf{p} \in S$,

for all $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}S$.

This is what we mean when we say that "the dot product is *preserved*" by a local isometry from S and $\tilde{S}$. We can also say that S and $\tilde{S}$ are _________________________.

Theorem 6.4. The map $f : S \to \tilde{S}$ is a local isometry *iff* it preserves the first fundamental form.

Proof. Let S be parametrized by $\boldsymbol{\sigma}(u, v)$. For example, note that on S, the coefficient $E =$____________, whereas on $\tilde{S}$, $\tilde{E} =$ ____________.

If f is a local isometry. We want to show that S and $\tilde{S}$ have the same first fundamental form. Since f preserves the dot product, we see that:

Thus, f preserves the first fundamental form.

Conversely, if f preserves the first fundamental form, we want to show that

Take any $\mathbf{w} \in T_{\mathbf{p}}S$. We can write $\mathbf{w}$ as a linear combination of _____ and _____ as:

This means that $|\cdot| = |\cdot|_f$, *i.e.* f preserves the norm. But the preservation of the norm is equivalent to the preservation of the dot product due to Lemma 6.3.

Hence, f a local isometry. □

Recall from the beginning of the Chapter that the first fundamental form determines the length of curves on surfaces. Hence, we immediately have the following result from the previous theorem.

Corollary. A map is a local isometry *iff* it preserves the length of curves.

So far, we have proved that the following statements are all equivalent.

- f preserves the ________________________ on $T_{\mathbf{p}}S$ and on $T_{f(\mathbf{p})}\tilde{S}$.
- f preserves the ________________________________.
- f preserves the ________________________ on S and on $\tilde{S}$.

These results give us several ways to check whether two surfaces are locally isometric. It's usually easiest to work with the first fundamental form.

Corollary. The cylinder (Example 4) and the plane (Example 2) are locally isometric.

Example 5. Show that the surface $\boldsymbol{\sigma}(u,v) = (u\cos v, u\sin v, u)$ with $0 < u < 1$ and $0 < v < 2\pi$, is locally isometric to the surface

$$\tilde{\boldsymbol{\sigma}}(\tilde{u},\tilde{v}) = \left(\sqrt{2}\tilde{u}\cos\frac{\tilde{v}}{\sqrt{2}}, \sqrt{2}\tilde{u}\sin\frac{\tilde{v}}{\sqrt{2}}, 0\right), \quad \text{with } \ 0 < \tilde{u} < 1, \ 0 < \tilde{v} < 2\pi.$$

Interpret this result.

6.4.1 The meaning of *local*

Given the length-preservation property of a local isometry, it's easy to gather the meaning of *isometry*. But what exactly is *local* about it?

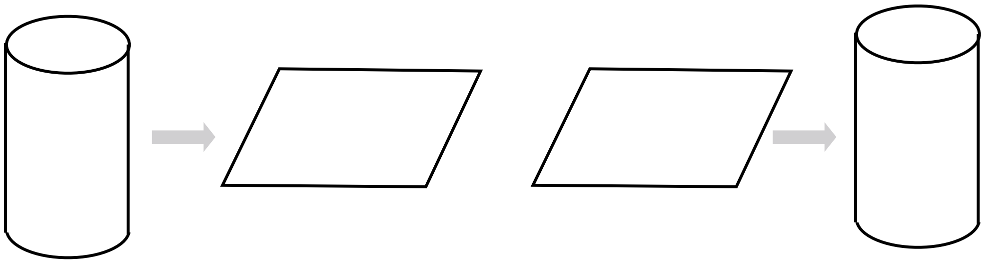
The plane and the cylinder are locally isometric. Intuitively, this means that, a small piece of a cylinder looks like a plane to its inhabitant.

But globally there is a difference – you can't smoothly transform a cylinder to a plane without cutting it open – one has a hole and one doesn't!

Definition. A global *isometry* is a local isometry which is bijective.

If S and $\tilde{S}$ are isometric surfaces (we normally drop the 'global'), all continuous paths between two points on S are mapped to those on $\tilde{S}$ (and vice versa) in a 1-to-1 correspondence, and all these corresponding paths have the same length (thanks to the length-preservation property). In particular, the *shortest* part between two given points in S is also mapped to the shortest path in $\tilde{S}$.

We can now see heuristically from the diagrams below why the cylinder can't be *globally* isometric to the plane. Suppose we have a isometry f : cylinder $\to$ plane. We see that the shortest distance between 2 points is not necessarily preserved.



This is because cutting and gluing are not bijections. They are not invertible. They are not homeomorphisms.

But there is no such problem if the two points are within a sufficiently small neighbourhood. In this sense, the isometry between these surfaces is only a *local* one.[1]

[1] Another way to understand the word *local* is via the inverse function theorem: a smooth map from S to $\tilde{S}$ is bijective in some neighbourhood of $\mathbf{p} \in S$ if the Jacobian of the mapping is nonzero. Thus, a local isometry is an isometry in a local neighbourhood.

6.4.2 What does all this mean for Anton the ant?

Anton the ant lives on a piece of A4 paper. He has studied the geometry of his world by measuring lengths of between places, angles between lines, area of figures *etc.* and has written a book compiling all his measurements. He is unaware of, and unable to explore, the 3D space outside the surface.

One day, strong wind blows the piece of paper on which Anton lives, landing it in a strange deformed configuration (luckily it's not creased, torn or perfectly rolled into a tube).

The wind has now died down, and Anton then sets about remeasuring all the results that he compiled in his book. What would he now find?

He will find that *all his measurements are still exactly the same.* In this Chapter, we have shown that his measurements of lengths will be unchanged. In Chapter 7, we will show that his measurements of *angles* and *areas* are also unchanged.

A surface inhabitant like Anton who can only measure *intrinsic* geometry of the surface (*e.g.* length, angle, area) is unable to decide whether he lives on a plane or any of the shapes shown below, which are all locally isometric to the plane. This is true despite the fact that to an observer outside the surface, it is easy to distinguish between them.

In fact, Anton is even able to measure a kind of intrinsic *curvature* of his new world, called the *Gaussian curvature*, which is unchanged despite the deformation of the paper. This is the famous *Theorema Egregium* ('remarkable theorem') of Gauss. More about this in Chapter 8.

Chapter 7

Conformal and Equiareal maps

A (local) isometry preserves length on different surfaces. In this Chapter, we will study special maps that preserve angle or area on surfaces.

7.1 Angle-preserving maps

Suppose that we have two curves $\boldsymbol{\gamma}_1(t)$ and $\boldsymbol{\gamma}_2(t)$ on surface S parametrized by $\boldsymbol{\sigma}(u, v)$. If the curves intersect at $\mathbf{p} = \boldsymbol{\gamma}_1(0) = \boldsymbol{\gamma}_2(0)$ (we can always rescale t so that this happens), what is the angle at which the curves intersect?

This is simply the angle, θ, between the *tangent vectors* of the curves evaluated at $\mathbf{p}$. Thus, calculating the angle between curves reduces to calculating the angle between vectors on $T_{\mathbf{p}}S$. But writing these tangent vectors (say, $\mathbf{v}$ and $\mathbf{w}$) as linear combinations of _____ and _____, we see that

and thus the problem further reduces to finding the angle between those basis vectors. In fact, we have tried this in Chapter 5 (Example 4) for surfaces of revolution. We will generalise that result in this Chapter.

Lemma 7.1. Let S be parametrized by $\boldsymbol{\sigma}(u, v)$ and $\mathbf{p} \in S$. The angle, θ, between $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ on $T_{\mathbf{p}}S$ can be expressed in terms of the first-fundamental-form coefficients as

$$\cos\theta =$$

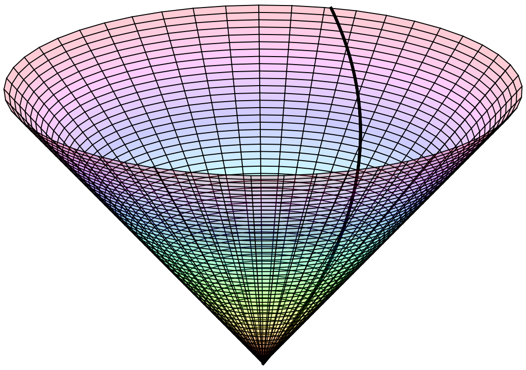
Proof:

Example 1. Calculate the angle between the base curve and the rulings of the ruled surface

$$\boldsymbol{\sigma}(u, v) = (\cos u - v \sin u, \;\; \sin u + v \cos u, v).$$

Sketch the surface and indicate this angle on your sketch.

Example 2. Recall when Anton the ant walked on the cone in Chapter 6 – he walked along the path $u = v = t$ from $t = 0$ to $t = 1$. Calculate the angle that his trajectory makes with the rim of the cone.



Let's now consider a map, $f : S \to \tilde{S}$, between 2 surfaces. The curves $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ (intersecting at angle θ on S) are mapped onto two intersecting curves _________ and _________. The new angle of intersection could potentially be different from θ.

Again this problem reduces to comparing two angles:

- the angle, θ, between vectors $\mathbf{v}$ and $\mathbf{w}$ on $T_{\mathbf{p}}S$
- the angle, $\tilde{\theta}$, between

Definition. A diffeomorphism[1] $f : S \to \tilde{S}$ is said to be a ________________________ if $\theta = \tilde{\theta}$ (as defined above). In other words, f is conformal if, for all $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}S$,

$$= \tag{7.1}$$

It's easy to remember this definition in words:

Theorem 7.2. The diffeomorphism $f : S \to \tilde{S}$ is a conformal map *iff* there exists a function $\lambda : S \to \mathbb{R}$, with $\lambda(\mathbf{p}) > 0$ for all $\mathbf{p} \in S$, such that

$$\tag{$\star$}$$

for all $\mathbf{p} \in S$ and $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}S$.

[1] we require f to be a diffeomorphism so that regular curves are mapped onto regular curves, *i.e.* their tangents don't become degenerate. See exercise 4.4.4 in Pressley for details.

Proof. Suppose ($\star$) holds.

To prove the converse, see problem sheet 7 for a step-by-step guide.

$\square$

By letting $\mathbf{v}$ and $\mathbf{w}$ be $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ in ($\star$), we immediately have the following:

Corollary. f is a conformal map *iff* the first fundamental forms of S and $\tilde{S}$ are related by

Example 3. Prove that all local isometries are conformal maps.

Example 4. Show that the surface parametrized by

$$\boldsymbol{\sigma}(u,v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u) \tag{7.2}$$

is conformal to a plane. What surface is this?

Definition. A surface S is said to be ________________________ if its first fundamental form satisfies

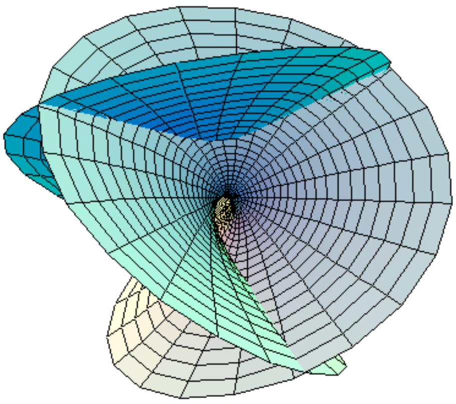
$$\mathrm{d}s^2 = \lambda^2(u,v)(\,\mathrm{d}u^2 + \,\mathrm{d}v^2).$$

Conformally flat surfaces are particularly interesting because angle measurements alone (*i.e.* trajectories of light rays, trajectories of ships sailing on the surface...) will not be able to distinguish the surface from a plane.

Example 5. *Enneper's surface* is parametrized by

$$\boldsymbol{\sigma}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right).$$

Show that $E = (1 + u^2 + v^2)^2$. Is it conformally flat?



7.2 Area-preserving maps

Consider a surface parametrized by $\boldsymbol{\sigma}(u, v)$. What is the area on the surface over some given range of u and v?

Let's divide the area into small parallelograms as shown, and consider the vectors representing the two sides of the parallelogram.

Proposition 7.3. The area of the portion of the surface parametrized by $\boldsymbol{\sigma}(u, v)$ corresponding to the domain where $(u, v) \in R$ is given by

(7.3)

This formula for area is unchanged with surface reparametrization[2], just as the arc-length is unchanged with curve reparametrization.

[2] see for Prop. 6.4.3 in Pressley.

Example 6. Using spherical coordinates, calculate the surface area of the unit sphere.

The area formula (7.3) could be used as it is, but it's often easier to express it in terms of the first-fundamental-form coefficients. The resulting formula has no cross products.

Definition. The *metric tensor* of a parametrized surface is defined as the matrix

$$g \equiv \tag{7.4}$$

Proposition 7.4. $|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v| =$ __________

Proof. Recall the following formula from vector calculus

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}),$$

(you should try proving this using the ε_{ijk} technique). Now substitute ...

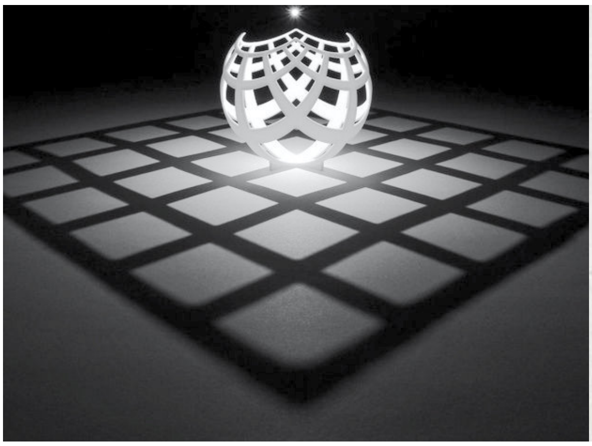
Lemma 7.5. The area of the surface $\boldsymbol{\sigma}(u,v)$ with metric g is given by

$$\tag{7.5}$$

Example 7. Recalculate the surface area of the sphere using the first fundamental form.

Definition. A map $f : S \to \tilde{S}$ is said to be ______________ if it maps any region in S to a region of the same area in $\tilde{S}$.

For example, the stereographic projection from the unit sphere to $\mathbb{R}^2$ (see sheet 4) is clearly *not* an equiareal map, as shown in the pictures[3] below. In sheet 7 you'll show that it is a conformal map.



[3] *Left:* see Henry Segerman's channel on YouTube for more 3D printed mathematical objects. *Right:* Rubens (1577-1640) was commissioned to draw this for the cover of a textbook on optics.

We saw in Theorem 6.4 that a map is length-preserving *iff* it preserves g. The next theorem shows that a map is *area*-preserving *iff* it it preserves __________.

Theorem 7.6. Let $f : S \to \tilde{S}$ be a diffeomorphism, with S and $\tilde{S}$ parametrized by $\boldsymbol{\sigma}(u, v)$ and $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$. Then, f is equiareal *iff*

$$\tag{7.6}$$

where $E = \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_u \rangle$ and $\tilde{E} = \langle \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_u \rangle_f$, *etc.* .

Proof. If (7.6) holds, then the area is clearly preserved. Conversely, if

$$\iint_R \sqrt{\det g}\ \mathrm{d}u\,\mathrm{d}v = \iint_R \sqrt{\det \tilde{g}}\ \mathrm{d}u\,\mathrm{d}v$$

for all regions R in the domain, then we must have $\det g = \det \tilde{g}$. See problem sheet 7 for details on this last step. □

7.2.1 Cylindrical projection

Have you ever noticed that the surface area of a sphere equals that of the cylinder which exactly contains it?

We now study one of the most famous equiareal maps, attributed to Archimedes who discovered this in $\sim$ 250BC.

Archimedes showed, using rudimentary geometrical arguments (remember this is pre-calculus era), that any horizontal strip on the sphere has the same area as the same strip on the cylinder which exactly contains the sphere.

Archimedes' tombstone (now lost) was said to be inscribed with a sphere contained within a cylinder. Here we study a proof of his result using differential geometry.

Consider the *cylindrical projection*, f, mapping the sphere to the cylinder where

$$\begin{aligned}\boldsymbol{\sigma}(u,v) &= (\sqrt{1-u^2}\cos v, \sqrt{1-u^2}\sin v, u), \quad u \in \\ \tilde{\boldsymbol{\sigma}}(u,v) &= (\cos v, \sin v, u), \quad u \in\end{aligned}$$

(I've dropped the pesky tildes). Note that the z coordinates of the sphere and cylinder are the same.

Theorem 7.7. (*Archimedes' Theorem*) The cylindrical projection is equiareal.
Proof:

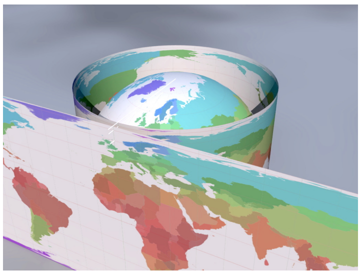
$\square$

Thus we can choose any region, R, on the sphere and project it horizontally to a region, $\tilde{R}$ on the wrapping cylinder. R and $\tilde{R}$ will have the same area.

In particular, the patches we chose cover the entire sphere and cylinder (with straight lines removed – they have zero area anyway), so the entire sphere and the cylinder have the same surface area.

However, note that this map is clearly not isometric nor conformal.

The cylindrical projection can be used in *cartography* (map-making). The resulting map, called the *Lambert cylindrical projection* of the Earth, is one which does not distort areas (unlike the popular *Mercator* projection which we will study in problem sheet 7). However, the Lambert projection distorts lengths and angles.



Question. Is there a map of the Earth that preserves both area and angle?

We'll come back to this question in the next (and final) Chapter.

7.3 Summary

Chapters 6 and 7 deal with special kinds of mapping between surfaces: maps that preserve lengths, angles or areas. The table below summarises the main results from these Chapters.

Name	Defining property	What's preserved?

Chapter 8

Curvature of surfaces

8.1 Motivation

How do we measure the curvature of a surface? Recall that for a plane curve, $\boldsymbol{\gamma}(t)$, its curvature is completely described by the signed curvature, κ_s, which, according to problem sheet 2, satisfies the equation

$$\dot{\mathbf{n}}_s = -\kappa_s \mathbf{t} \implies$$

where $\mathbf{t}$ is the tangent vector and $\mathbf{n}_s$ is the ___________________ (see §2.2). Loosely, we can say that for a plane curve, its signed curvature is the (negative of the) rate of change of the normal vector ______________ in the direction of the tangent vector. It makes intuitive sense that a curve whose normal changes rapidly seems to curve *more* than one whose normal does not change much.

We could try to extend this idea to quantify how much a surface curves. Recall from sheet 5 that the standard unit normal of a surface $\boldsymbol{\sigma}(u, v)$ is defined as

$$\hat{\mathbf{N}} \equiv$$

The "rate of change" of this quantity is captured by two *partial* derivatives: _______ and _______. We then project (or dot) them in the direction of some unit tangent vector, which, for surfaces, is a linear combination of _______ and _______.

Thus, the quantities that could help us measure the curvature of a surface are:

(8.1)

(actually we will show later that two of them are equal). These quantities are called the *second fundamental form*. We will show that the first *and* second fundamental forms together completely capture the information on the curvature of a surface.

8.2 The Gauss map

Let's examine the unit normal $\hat{\mathbf{N}}$ more carefully.

Any unit vector can be thought of as a vector joining the origin to a point on the unit sphere, and $\hat{\mathbf{N}}$ is no exception. In this sense, we can think of $\hat{\mathbf{N}}$ is a *map* from a point $\mathbf{p}$ on a surface S, to a point $\hat{\mathbf{N}}(\mathbf{p})$ on the unit sphere.

We will be talking about the unit sphere a lot so let's refer to it by the standard nickname[1] $\mathbb{S}^2$, *i.e.*

(8.2)

Definition. The *Gauss map*, $\hat{\mathbf{N}} : S \to \mathbb{S}^2$, is defined as follows: $\forall \mathbf{p} \in S$, $\hat{\mathbf{N}}(\mathbf{p})$ is the standard unit normal at $\mathbf{p}$.

[1] $\mathbb{S}^1$ is the unit circle. $\mathbb{S}^n$ is defined by the equation $x_1^2 + x_2^2 + \ldots x_{n+1}^2 = 1$

Example 1. No calculations allowed. Describe the image of the Gauss map of
a) The unit sphere, b) the unit cylinder, c) a plane.

In the above examples, we have assumed that the normals are all pointing upwards or outwards (of course this will depend on the parametrization), but check that the answers are unchanged had we assumed the opposite directions.

Note that the concepts of in/out/up/down do not make sense on non-orientable surfaces like the ____________________. In this Chapter we will only deal with surfaces that are orientable.

Example 2. Calculate the Gauss map $\hat{\mathbf{N}}$ of the surface $z = x^2 + y^2$. Describe the image of $\hat{\mathbf{N}}$.

It's worth clarifying our notation at this point. $\hat{\mathbf{N}}$ can be regarded as either a *vector*, or a *map*. We will not distinguish our notation as it should be clear from the context in what form we are regarding $\hat{\mathbf{N}}$ as.

Here's a nice observation before we proceed. When regarded as a unit vector, we have the following lemma which should be immediately clear.

Lemma 8.1. For a regular surface $\boldsymbol{\sigma}(u,v)$, $\{\hat{\mathbf{N}}, \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ form a basis of $\mathbb{R}^3$. If the first-fundamental form coefficient $F = 0$, the basis is *orthogonal*. If, in addition, $E = G = 1$, then the basis is *orthonormal*.

Can you think of an analogous statement for curves in $\mathbb{R}^3$?

8.3 The Weingarten map

The Gauss map takes a surface S to (part of) $\mathbb{S}^2$. How are their tangent spaces related? Recall that the mapping between two tangent spaces is just the ______________.

$$: T_{\mathbf{p}}S \to T_{\hat{\mathbf{N}}(\mathbf{p})}\mathbb{S}^2. \tag{8.3}$$

The tangent plane $T_{\hat{\mathbf{N}}(\mathbf{p})}\mathbb{S}^2$ contains all vectors perpendicular to the normal $\hat{\mathbf{N}}(\mathbf{p})$, which, by definition, is just __________. Thus the derivative of $\hat{\mathbf{N}}$ can be regarded as a map from $T_{\mathbf{p}}S$ onto itself:

$$\tag{8.4}$$

Let's see exactly how this derivative looks like.

Theorem 8.2. Let $\mathbf{p} \in S$. The derivative of the Gauss map satisfy

$$\begin{aligned} D_{\mathbf{p}}\hat{\mathbf{N}}(\boldsymbol{\sigma}_u) &= \hat{\mathbf{N}}_u \\ D_{\mathbf{p}}\hat{\mathbf{N}}(\boldsymbol{\sigma}_v) &= \hat{\mathbf{N}}_v \end{aligned} \tag{8.5}$$

(In other words, for the Gauss map, the concept of the derivative of a map reduces to the good old partial derivative.)

Proof: Here are some facts from Chapter 6: Any $\mathbf{w} \in T_{\mathbf{p}}S$ can be thought of as a tangent vector to some curve $\boldsymbol{\gamma}(t)$, evaluated at say, at $t = 0$

$$\mathbf{w} = \tag{8.6}$$

By definition, the derivative $D_{\mathbf{p}}\hat{\mathbf{N}}$ gives the tangent of the mapped curve (Eq. 6.7).

$$D_{\mathbf{p}}\hat{\mathbf{N}}(\mathbf{w}) \equiv \tag{8.7}$$

Pick a point $\mathbf{p} = \boldsymbol{\sigma}(u_0, v_0)$. Consider the curve $\boldsymbol{\gamma}(u) = \boldsymbol{\sigma}(u, v_0)$ (v=constant curves) undergoing the mapping $\hat{\mathbf{N}}$. The image of the tangent vector $\boldsymbol{\gamma}'(u) = \boldsymbol{\sigma}_u(u, v_0)$ is, by definition, the derivative of $\boldsymbol{\sigma}_u$. In symbols:

$$D_{\mathbf{p}}\hat{\mathbf{N}}(\boldsymbol{\sigma}_u) = \tag{8.8}$$

The RHS is an overly complicated way to express what is simply ________.

The proof for $\boldsymbol{\sigma}_v$ is similar. □

There's a special name for the (negative of) derivative of the Gauss map.

Definition. The *Weingarten*[2] *map* $W : T_{\mathbf{p}}S \to T_{\mathbf{p}}S$ is defined as

$$\tag{8.9}$$

For example, the previous theorem can be expressed in terms of the Weingarten map as:

$$\tag{8.10}$$

[2] *Julius Weingarten*(1836-1910), German mathematician who made important contributions to differential geometry. Note that in some books, the Weingarten map is called the *shape operator*.

The minus sign is just a historical convention, just like the minus sign in $\kappa_s = -\dot{\mathbf{n}}_s \cdot \mathbf{t}$.

Example 3. Prove that

(a) $\hat{\mathbf{N}}_u \cdot \hat{\mathbf{N}} = \hat{\mathbf{N}}_v \cdot \hat{\mathbf{N}} = 0$.

(b) $\hat{\mathbf{N}}_u \cdot \boldsymbol{\sigma}_u = -\hat{\mathbf{N}} \cdot \boldsymbol{\sigma}_{uu}$ and find 3 other similar results.

Deduce that $\hat{\mathbf{N}}_u \cdot \boldsymbol{\sigma}_v = \hat{\mathbf{N}}_v \cdot \boldsymbol{\sigma}_u$.
Why is there no need to distinguish between the dot product and that 'induced' by $\hat{\mathbf{N}}$?

8.4 The second fundamental form

Our train of thought so far:

- To quantify 'curvature' at a point on a surface, we measure the rate of change of the unit normal vector $\hat{\mathbf{N}}$, projected in some tangential direction (taking the queue from the formula $\kappa_s = -\dot{\mathbf{n}}_s \cdot \mathbf{t}$).
- The analogues of $\dot{\mathbf{n}}_s$ are $\hat{\mathbf{N}}_u$ and $\hat{\mathbf{N}}_v$.
- The analogue of $\mathbf{t}$ is any vector $\mathbf{w} \in T_{\mathbf{p}}S$, which is a linear combination of $\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v$.
- Thus, curvature could be measured by the four quantities:
- Alternatively, from Example 3, these quantities look like:
- In terms of the *Weingarten map*, W, these quantities look like:

Thus, at point $\mathbf{p}$ with tangent vector $\mathbf{w} \in T_{\mathbf{p}}S$, the expression ______________ contains all four curvature quantities at that point. We now show that $W(\mathbf{w}) \cdot \mathbf{w}$ can be expressed in a similar way to the first fundamental form.

Lemma 8.3. Let $\mathbf{p}$ be a point on surface S parametrized by $\boldsymbol{\sigma}(u, v)$. For any $\mathbf{w} \in T_{\mathbf{p}}S$, let $\mathbf{w}$ be the tangent to a curve $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ passing through $\mathbf{p}$. We have

$$W(\mathbf{w}) \cdot \mathbf{w} = L \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + 2M \frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}t} + N \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 \tag{8.11}$$

where $L =$ ______________, $M =$ ______________, $N =$ ______________.

Proof:

Equation 8.11 can be written in a similar way to the first fundamental form by defining

$$II \equiv W(\mathbf{w}) \cdot \mathbf{w}\ \mathrm{d}t^2 \tag{8.12}$$

Definition. The *second fundamental form* of the surface parametrized by $\boldsymbol{\sigma}(u, v)$ is defined as the expression.

$$II = \tag{8.13}$$

Information on the curvature of a surface is (partly) contained in the second fundamental form. Exactly how to fully extract this information remains to be seen...

Example 4. Calculate the second fundamental form of the following surfaces:

(a) $\boldsymbol{\sigma}(u, v) = \mathbf{a} + u\hat{\mathbf{p}} + v\hat{\mathbf{q}}$ (where $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ are orthonormal vectors)

(b) $\boldsymbol{\sigma}(u, v) = (\cos v, \sin v, u)$.

8.5 The Weingarten map in matrix form

Suppose Anton lives on the unit cylinder, and we have asked him to measure how much the cylinder curves at a point $\mathbf{p}$ on the cylinder. You can see that the answer depends on the route that Anton takes across $\mathbf{p}$.

Thus, we expect that there will be several definitions of surface curvature. We will now show that all types of curvatures can be fully extracted from the *Weingarten map*, which can be expressed as an elegant matrix. This is where the first and second fundamental forms come together in a grand conclusion!

Theorem 8.4. Let a surface S be parametrized by $\boldsymbol{\sigma}(u,v)$. The Weingarten map W can be expressed in matrix form as

$$W =$$

wrt the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$, where g and $\mathcal{F}_{II}$ are matrices of the coefficients of the first and second fundamental forms:

$$g = \qquad\qquad\qquad \mathcal{F}_{II} =$$

Proof. From Eq. 6.9, we know that $W : T_{\mathbf{p}}S \to T_{\mathbf{p}}S$ is a linear map (linearity inherited from the derivative), and can be represented by a matrix multiplication. To express W in the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$, let's see how it acts on each of the basis vectors. Suppose that

$$\begin{aligned} W\boldsymbol{\sigma}_u &= \qquad (*) \\ W\boldsymbol{\sigma}_v &= \end{aligned}$$

for some numbers a, b, c, d. With respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$, the above equations can be written as:

$$W \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

On the other hand, dotting equations $(*)$ with $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ yields four equations.

$\square$

Example 5. Find the Weingarten map W (in matrix form) for the plane and the cylinder in Example 4.

Example 6. Find a general expression for the matrix W in terms of E, F, G, L, M, N.

8.6 Types of curvatures

We have shown that information about the curvature can be represented by a matrix W which looks nice wrt the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$. What if we choose a different basis? Whatever quantity which represents the physical concept of "curvature" shouldn't depend on the basis in which the matrix is expressed.

What properties of a matrix are unchanged by a change of basis? Linear Algebra tells us that they are:

These are all good candidates for capturing the essence of surface curvature.

Definition. The *Gaussian curvature*, K, is defined as

(8.14)

Definition. The *mean curvature*, H, is defined as

(8.15)

Definition. The *principal curvatures*, κ_1 and κ_2, are defined as the ____________________ of W

These are not all independent quantities. Since K and H are invariant wrt to change of basis, we could diagonalise W and obtain the following result.

Corollary. The mean and Gaussian curvatures are related to the principal curvatures as follows:

Example 7. Find the Gaussian, mean and principal curvatures for the plane and the unit cylinder.

The previous Example gave us a glimpse into the meaning of these curvatures.

For instance, the cylinder has one circular direction ($\kappa = 1$) and one flat direction ($\kappa = 0$), so it makes sense that the *mean* curvature is indeed $1/2$.

The cylinder can be thought of as a rolled up plane, and $K = 0$ on both objects. This tells us that the Gaussian curvature is an *intrinsic* property of a surface that is invariant under local isometry (*i.e.* length-preserving transformations like bending, rolling, twisting). This is a remarkable result that we will discuss in more detail shortly.

Example 8. Express H and K in terms of the fundamental-form coefficients E, F, G, L, M, N.

8.7 Concluding remarks

By now you should have a good idea of what *Differential Geometry* is all about. We have seen how geometrical properties of curves and surfaces can be revealed using a combination of Calculus, Linear Algebra and Analysis.

Let me wrap up this course with a few remarks to tie up loose ends, and to give you a glimpse of what lies beyond this course. The results in this section are non-examinable.

8.7.1 Area interpretation of the Gaussian Curvature

Theorem 8.5. Suppose an area element $\mathrm{d}A$ on a surface S is mapped onto an area $\mathrm{d}\tilde{A}$ on the unit sphere via the Gauss map, $\hat{\mathbf{N}}$, then

$$|K|\ \mathrm{d}A = \mathrm{d}\tilde{A}$$

Proof. The area element on S is given by

$$\mathrm{d}A =$$

The Gauss map sends this area to $\mathrm{d}\tilde{A}$ on the unit sphere, where

$$\mathrm{d}\tilde{A} =$$

From the proof of Theorem 8.4, we find the following expressions for $\hat{\mathbf{N}}_u$ and $\hat{\mathbf{N}}_v$:

Amazingly, the Gaussian curvature is just a ratio of area elements post- and pre-Gauss map. From this result, you can see why $K = 0$ for the cylinder and plane: the area element post-Gauss map is zero (see the drawings we made in Example 1).

8.7.2 Gauss-Bonnet Theorem

Recall *Hopf's Umlaufsatz* for curves:

The signed curvature is a 'local' quantity (*i.e.* defined at a point on a surface), but the *total* signed curvature is clearly a *global* property of closed curves.

A similar link exists for surfaces. The Gaussian curvature K is a local property but the total Gaussian curvature.

is a global property of 'closed' (also known as *compact*) surfaces. If we assume that the image of the Gauss map for compact surfaces covers the entire unit sphere, so that we roughly have

(8.16)

assuming something like the result in the previous section.

Eq. 8.16 is a special case of one of the most beautiful theorems in differential geometry: the *Gauss-Bonnet*[3] *Theorem*, which says:

Theorem 8.6. [*Gauss-Bonnet Theorem*] Let S be a compact surface, then

$$=$$

where $\chi = 2 - 2g$, and g is the __________ (the number of holes) on the surface.

The surface constant χ is called the ______________________________.

[3] *Pierre Ossian Bonnet* (1819-1892), French mathematician famed for his contribution to differential geometry. He was a contemporary of Frenet and Serret.

The Gauss-Bonnet theorem holds not only for 'smooth' surfaces, but also for solids such as a cube or a dodecahedron (things with vertices and edges). This single theorem is the bridge that connects differential geometry to *topology* ('rubber-sheet geometry').

8.7.3 Theorema Egregium

Every mathematician knows that the German mathematician *Carl Friedrich Gauss* (1777-1855) ranks amongst the top mathematical minds *ever.* Amongst his many illustrious and wide-ranging achievements, his 'remarkable theorem' (as he himself called it) is indeed one of the most remarkable. We state it here without proof.

Theorem 8.7. [*Gauss's Theorema Egregium*] The Gaussian curvature, K, can be expressed solely in terms of the first-fundamental-form coefficients E, F, G and their derivatives.

This means that *local isometries do not change Gaussian curvature*, *i.e.* one can regard K is a 'bending invariant' which is an *intrinsic* property of surface, like its area, its thickness or its density. Anton, for instance, will find that his measurement of K is the same before and after the incident at the end of Chapter 6.

In fact, K can be written explicitly in terms of E, F, G as

$$K = \frac{1}{(EG-F^2)^2}\left(\begin{vmatrix} -\frac{1}{2}E_{vv}+F_{uv}-\frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u-\frac{1}{2}E_v \\ F_v-\frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}\right). \tag{8.17}$$

This is sometimes called the 'Brioschi formula'. You can quickly verify that it works for the plane and cylinder.

8.7.4 A perfect map doesn't exist

Here is the final word on cartography, courtesy of the *Theorema Egregium.*

Lemma 8.8. There exists no geographical map of any portion of the Earth which is both equiareal and conformal.

Proof. Suppose there exists such a map. By Q1 of problerm sheet 7, this map is a ___________________. This means the first fundamental forms are the same on the sphere and the plane. The *Theorema Egregium* implies that K is the same on these surfaces. However, $K =$ _____ on the plane, and $K > 0$ on the sphere (see problem sheet 8). A contradiction. □

8.7.5 The Fundamental Theorem of Surfaces

Analogous to the Fundamental Theorem of Curves, we have the Fundamental Theorem of Surfaces (due to Bonnet).

Theorem 8.9. [Fundamental Theorem of Surface Theory] Let E, F, G, L, M, N be any smooth real-valued functions of u and v.

[Existence] There exists a regular, orientable surface S, parametrized by $\boldsymbol{\sigma}(u, v)$ with the first and second fundamental forms given by

$$E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 \quad \text{and} \quad L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2.$$

[Uniqueness] The surface is unique up to translation and rotation.

8.7.6 Looking ahead

Many results we study in this course can be generalised to more than 3D. A *manifold* is a topological space which looks like $\mathbb{R}^n$ locally (analogy: a surface looks like a plane locally). Differential Geometry on higher-dimensional manifolds is the most fundamental tool in the studies of general relativity, particle physics, string theory and cosmology.

Interesting topics for further studies in an advanced course on differential geometry (currently offered as a reading course in 4th year) include *geodesics* (shortest routes on surfaces), *minimal surfaces* (what shape does a soap bubble form?) and *non-Euclidean geometry* (could parallel lines diverge?).